KNIFE-EDGE CONDITIONS AND THE MACRODYNAMICS OF SMALL OPEN ECONOMIES

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1. Introduction

The application of rigorous macroeconomic dynamic models to small open economies has been a thriving area of research for nearly two decades, with these applications extending over a diverse range of issues. Inevitably, such models are characterized by special "knife-edge conditions," by which we mean that certain parameters, or combinations of parameters, are constrained to take on certain specific values in order for a viable equilibrium to exist.

This paper examines the macrodynamic structure of a small open economy and considers the role played by various standard knife-edge conditions. As we shall see, the restrictions imposed play an important part in determining the underlying equilibrium dynamics, and thus how the economy responds to alternative structural and policy shocks. Particularly in a growing economy, different knife-edge conditions lead to fundamental differences in the determination of long-run equilibrium growth rates, and in the scope for macroeconomic policy to influence them.

We shall lay out a generic dynamic model of a small open economy in sufficient generality so as to provide a unifying framework within which various classes of models can be placed. Three important classes of models are identified as special cases of this generic structure, thereby enhancing our understanding of the models themselves and the relationships between them. These models include: (i) the traditional stationary Ramsey model; (ii) the endogenous growth model; (iii) the non-scale growth model.

We shall consider three margins along which "knife-edge" restrictions are typically imposed. These include (i) the preference parameters of agents, (ii) production and employment characteristics, and (iii) openness of international financial markets. Moreover, important tradeoffs between these knife-edge conditions are shown to exist. Thus, for example, imposing restrictions on preferences permits greater flexibility for technology, and vice versa. We shall also focus on those knife-edge conditions that are critical to the implied macrodynamic equilibrium structure. There are in fact many other restrictions that one could also characterize as being knife-edge, but which, as in any economic theory, are imposed purely for the purposes of analytical tractability.

The three classes of models we shall consider span most of the models that have dominated international macroeconomics during the last two decades. The stationary Ramsey model was the first model to derive the macrodynamics of the international economy as the equilibrium outcome of intertemporal optimization by representative agents. As its name implies, it leads to a stationary equilibrium with a fixed, endogenously determined, stock of capital, and the cessation of all capital accumulation, except possibly to replace depreciated capital. The key knife-edge condition, compatible with the attainment of such an equilibrium, involves the equality between the world interest rate and the economy's rate of time preference, both of which most of these models take to be exogenously given constants. The stationary Ramsey model has been extensively employed to analyze the international effects of various types of real shocks, such as alternative commodities and income taxes, public expenditures, transfers, and supply shocks. A summary of the basic model, various extensions, and some examples of its applications is provided by Turnovsky (1997a).

Under knife-edge conditions on the underlying technology, the Ramsey model with fixed population generates an equilibrium of ongoing endogenous growth, thereby extending the basic Romer (1986)-Barro (1990), model to the open economy. These knife-edge conditions introduce fundamentally different behavior to that characterizing the Ramsey model. First, in the basic one-sector capital good model there are no transitional dynamics; instead, the economy always lies on its balanced growth path.[1] Second, the determination of this balanced-growth equilibrium depends much more critically upon whether or not the supply of labor is assumed to be fixed or endogenous than it does in the Ramsey model. With fixed labor supply, the equilibrium will sustain independent, but constant, growth rates for domestic consumption, on the one hand, and output and capital, on the other. The former is determined primarily by preference parameters and the foreign interest rate; the latter depends primarily upon domestic technological parameters, as well as the domestic income tax. With endogenous labor supply, all equilibrium growth rates are equal, and are driven by the determinants of the consumption growth rate. Changes in technology and in the income tax rate are reflected in the labor-leisure choice, rather than in the output growth rate.

[1] In order to generate transitional dynamics a second capital good must be introduced; see e.g. Bond, Wang, and Yip (1996) and Turnovsky (1997a).

The endogenous growth model has been subject to increasing criticisms on both empirical and theoretical grounds. Empirically, it is often associated with "scale effects," meaning that the equilibrium growth rate is related to the size of the economy (as measured by population, say), although this is unsupported by empirical evidence; see Jones (1995b), Backus, Kehoe, and Kehoe (1992). Second, the conditions on the technology necessary to obtain ongoing growth require that the production function have constant returns to scale in the endogenously accumulating factors, which is a very stringent knife-edge characteristic; see Solow (1994).

In response to these observations, Jones (1995a, 1995b), Segerstrom (1998), Young (1998), and others have proposed the so-called "non-scale" growth model.[2] This model relaxes the stringent returns to scale assumption associated with the endogenous growth model, and instead allows for both increasing and decreasing returns to scale in production. This leads to a very different macrodynamic structure from that of the endogenous growth model. First, the one-sector model is associated with transitional dynamics; second the equilibrium growth rate is independent of all fiscal instruments. But the non-scale model is not devoid of knife-edge conditions, either. In this case the nature of the labor supply is the critical issue. If labor is supplied inelastically, then no further restrictions are imposed. However, if labor is supplied endogenously, then equilibrium requires that consumption and output grow at the same rate, and to sustain this a further knife-edge condition, which is a direct generalization of that associated with the stationary Ramsey model, is needed.

The remainder of the paper is structured as follows. Section 2 sets out the general structure of the macrodynamic model of a small open economy. Sections 3 – 5 then consider (i) the Ramsy model, (ii) the endogenous growth model, and (iii) the non-scale growth model, as special cases. Most macrodynamic models of small open economies assume that the economy has unrestricted access to a perfect world capital market, and indeed this assumption is an important aspect of some knife-edge conditions. Section 6 briefly discusses the consequences of modifying this assumption for the macrodynamic equilibrium, while the main insights of our analysis are reviewed in Section 7.

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[2] Jones (1995a) originally described the model as being "semi-endogenous." As Jones (1999) has pointed out, two strands of these models are evolving. The basic model assigns no role for policy to influence long-run growth; more recent developments permit a limited role for government policy to influence growth through taxes and subsidies to research and development.

2. A Generic Model of a Small Open Economy

We begin by describing the generic structure of a small open economy in sufficient generality to enable us to identify the alternative special cases that correspond to alternative knife-edge characteristics. The economy we consider consumes and produces a single traded commodity, though the analysis extends to multi-sector economies. There are *N* identical individuals, each of whom has an infinite planning horizon and possesses perfect foresight. Each agent is endowed with a unit of time that can be allocated either to leisure, l, or to labor, $(1-l)$. Labor is fully employed so that total labor supply, equal to population, *N*, grows exponentially at the steady rate $\dot{N} = nN$. Individual domestic output, Y_i, of the traded commodity is determined by the individual's private capital stock, K_i, his labor supply, $(1-l)$, and the aggregate capital stock $K = NK_i$.[3] In order to accommodate growth under more general assumptions with respect to returns to scale, we assume that the output of the individual producer is determined by the Cobb-Douglas production function: [4]

$$Y_i = \alpha(1-l)^{1-\sigma} K_i^{\sigma} K^{\eta} \qquad 0 < \sigma < 1,\ \eta \gtrless 0 \tag{1a}$$

This formulation is akin to the earliest endogenous growth model of Romer (1986). The spillover received by an individual from the aggregate stock of capital can be motivated in various ways. One is to interpret *K* as knowledge capital, as Romer suggested. Another, is to assume *N* specific inputs (subscripted by *i*) with aggregate *K* representing an intra-industry spillover of knowledge.[5]

Each private factor of production has positive but diminishing marginal physical productivity. To assure the existence of a competitive equilibrium the production function must exhibits constant returns to scale in the two private factors [Romer (1986)]. In contrast to the

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[3]Since all agents are identical, all aggregate quantities are simply multiples of the individual quantities, $X = NX_i$. Note that since all agents allocate the same share of time to work, there is no need to introduce the agent's subscript to l.

[4]When production functions exhibit non-constant returns to scale in all factors, the existence of a balanced growth equilibrium requires the production function to be Cobb-Douglas, as assumed in (1a); see Eicher and Turnovsky (1999a). Note that the Cobb-Douglas function, being a special case of the CES production function, can be viewed as a knife-edge condition, as can the assumption of constant returns to scale in the private factors. However, these restrictions are mainly for convenience and are not essential to the determination of equilibrium. The stationary Ramsey model, to be discussed in Section 3, does not in general impose any functional form on the production function.

[5]A negative exponent can be interpreted as reflecting congestion, along the lines of Barro and Sala-i-Martin (1992).

standard neoclassical growth model, we do not insist that the production function exhibits constant returns to scale, and indeed total returns to scale are $1+\eta$, and are increasing or decreasing, according to whether the spillover from aggregate capital is positive or negative.

As we will show in subsequent sections, the production function is sufficiently general to encompass a variety of models. For example, we will show that the model is consistent with long-run stable growth, provided returns to scale are appropriately constrained. This contrasts with models of endogenous growth and externalities in which exogenous population growth can be shown to lead to explosive growth rates; see Romer (1990). We should also point out that the standard AK model emerges when $\sigma+\eta=1,\ n=0$, and the neoclassical model correspond to $\eta=0$.

Aggregate consumption in the economy is denoted by *C*, so that the per capita consumption of the individual agent at time *t* is $C/N=C_i$, yielding the agent utility over an infinite time horizon represented by the intertemporal isoelastic utility function:

$$\Omega \equiv \int_0^\infty (1/\gamma)\left(C_i l^\theta\right)^\gamma e^{-\rho t}dt; \quad -\infty<\gamma<1;\ \theta>0,\ 1>\gamma(1+\theta),\ 1>\gamma\theta \tag{1b}$$

where $1/(1-\gamma)$ equals the intertemporal elasticity of substitution, and θ measures the substitutability between consumption and leisure in utility. The remaining constraints on the coefficients in (1b) are required to ensure that the utility function is concave in the quantities *C* and *l*.

Agents accumulate physical capital, with expenditure on a given change in the capital stock, I_i, involving adjustment (installation) costs that we incorporate in the quadratic (convex) function

$$\Phi(I_i,K_i)\equiv I_i+h\,I_i^2/2K_i \quad = I_i\left(1+hI_i/2K_i\right)$$

This equation is an application of the familiar Hayashi (1982) cost of adjustment framework, where we assume that the adjustment costs are proportional to the *rate* of investment per unit of installed capital (rather than its level). The linear homogeneity of this function is necessary if a steady-state equilibrium having ongoing growth is to be sustained.[6] For simplicity we assume that the capital stock does not depreciate, so that the net rate of capital accumulation is given by:

[6] Many applications of the cost of adjustment in the Ramsey model assume that adjustment costs depend upon the absolute rate of investment, rather than its rate relative to the size of the capital stock. They also often assume only that

$$\dot{K}_i = I_i - nK_i \tag{1c}$$

In addition, the agents also accumulate foreign bonds, B_i, which pay a fixed rate of return, r, determined exogenously in the world bond market. We shall assume that income from current production is taxed at the rate τ_y, income from bonds is taxed at the rate τ_b, while in addition, consumption is taxed at the rate τ_c. We shall illustrate the contrasting implications of different models by analyzing the purely distortionary aspects of taxation and assume that revenues from all taxes are rebated to the agent as lump sum transfers T_i. Thus the individual agent's instantaneous budget constraint is described by:

$$\dot{B}_i = (1-\tau_y)Y_i + \left[r(1-\tau_b) - n\right]B_i - (1+\tau_c)C_i - I_i\left[1 + (h/2)(I_i/K_i)\right] + T_i \tag{1d}$$

The agent's decisions are to choose his rates of consumption, C_i, leisure, l, investment, I_i, and asset accumulation, B_i, K_i, to maximize the intertemporal utility function, (1a), subject to the accumulation equations, (1c) and (1d). The discounted Hamiltonian for this optimization is:

$$H \equiv e^{-\rho t}\frac{1}{\gamma}\left(C_i l^{\theta}\right)^{\gamma} + \lambda e^{-\rho t}\left[(1-\tau_y)Y_i - \Phi_i - (1+\tau_c)C_i + r(1-\tau_b)B - T_i - \dot{B}_i\right] + q' e^{-\rho t}\left[I - nK_i - \dot{K}_i\right]$$

where λ is the shadow value of wealth in the form of internationally traded bonds and q' is the shadow value of the agent's capital stock. Exposition of the model is simplified by using the shadow value of wealth as numeraire. Consequently, $q \equiv q'/\lambda$ can be interpreted as being the market price of capital in terms of the (unitary) price of foreign bonds.

The optimality conditions with respect to C_i, l, and I_i are respectively

$$C_i^{\gamma-1} l^{\theta\gamma} = \lambda(1+\tau_c) \tag{2a}$$

$$\theta C_i^{\gamma} l^{\theta\gamma-1} = \frac{\lambda(1-\tau_y)(1-\sigma)Y_i}{(1-l)} \tag{2b}$$

$$1 + h(I_i/K_i) = q \tag{2c}$$

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it is convex; the assumption of a quadratic function is made for convenience, simplifying the solution for the equilibrium growth rates in the endogenous growth model

Equation (2a) equates the marginal utility of consumption to the tax-adjusted shadow value of wealth, while (2b) equates the marginal utility of leisure to its opportunity cost, the after-tax marginal physical product of labor (real wage), valued at the shadow value of wealth. The third equation equates the marginal cost of an additional unit of investment, which is inclusive of the marginal installation cost $h\, I_i/K_i$, to the market value of capital. Equation (2c) may be solved to yield the following expression for the rate of capital accumulation:

$$\dot{K}_i/K_i = I_i/K_i - n = (q-1)/h - n \equiv \varphi_i \tag{3}$$

With all agents being identical, equation (3) implies that the growth rate of the aggregate capital stock, is $\varphi = \varphi_i + n$, so that

$$\dot{K}/K = \dot{K}_i/K_i + n = (q-1)/h \equiv \varphi \tag{3'}$$

This describes a "Tobin *q*" theory of investment, with $\dot{K} \gtrless 0$ according to whether $q \gtrless 1$.[7]

Optimizing with respect to B_i and K_i implies the arbitrage relationships

$$\rho - \left(\dot{\lambda}/\lambda\right) = r(1-\tau_b) - n \tag{4a}$$

$$\left((1-\tau_y)\sigma Y_i/qK_i\right) + \left(\dot{q}/q\right) + (q-1)^2/2hq = r(1-\tau_b) \tag{4b}$$

Equation (4a) is the standard Keynes-Ramsey consumption rule, equating the marginal return on consumption to the growth-adjusted after-tax rate of return on holding a foreign bond. Likewise (4b) equates the after-tax rate of return on domestic capital to the after-tax rate of return on the traded bond. The former comprises three components. The first is the marginal after-tax output per unit of installed capital, (valued at the price *q*), while the second is the rate of capital gain. The third element reflects the fact that an additional benefit of a higher capital stock is to reduce the installation costs (which depend upon I_i/K_i) associated with new investment. Finally, in order to

[7] In the case where the cost of adjustment is expressed in absolute terms, (3') determines the level of investment.

ensure that the agent's intertemporal budget constraint is met, the following transversality conditions must be imposed:[8]

$$\lim_{t\to\infty} \lambda B_i e^{-\rho t} = 0; \quad \lim_{t\to\infty} \lambda q K_i e^{-\rho t} = 0 \tag{4c}$$

The government in this economy plays a limited role. It levies income taxes on output and foreign interest income, it taxes consumption, and then rebates all tax revenues. In aggregate, these decisions are subject to the balanced budget condition

$$\tau_y Y + \tau_b rB + \tau_c C = T \tag{5}$$

Aggregating (1d) over the N individuals, and imposing (5) and (1c) leads to:

$$\dot{B} = Y + rB - C - I\left[1 + (h/2)(I/K)\right] \tag{6}$$

which describes the country's current account.

The key components of the model are the 5 optimality conditions (2a) –(2c), (4a), and (4b), and the critical issue is how the macrodynamic equilibria these generate are affected by the knife-edge conditions that characterize the different models. Note that the optimality condition for labor, (2b), ceases to be applicable when labor is supplied inelastically.

3. The Stationary Ramsey Model

The stationary Ramsey model is obtained from Section 2 by introducing the following special assumptions. First, the population is stationary, ($n = 0$), and without loss of generality can be set to unity ($N = 1$). Individual and aggregate quantities are therefore identical, and so the individual agent subscript can be dropped. Second, the traditional Ramsey model excludes external spillovers; that is, $\eta = 0$ so that the production function (1a) becomes:

$$Y = \alpha(1-l)^{1-\sigma} K^{\sigma} \tag{1a'}$$

[8]The transversality condition on debt is equivalent to the national intertemporal budget constraint.

The key knife-edge condition associated with this economy arises from the arbitrage condition (4a). With r, τ_b, ρ all being constant through time, (4a) implies that the marginal utility of wealth and therefore consumption will rise or fall indefinitely unless the after-tax return on foreign bonds equals the rate of time preference

$$r(1-\tau_b) = \rho \tag{7}$$

But (7) has strong implications for the macroeconomic equilibrium. First, if (7) holds then $\dot{\lambda} = 0$, not only in stationary equilibrium, but at all points of time. This implies that the marginal utility of wealth is constant through time, $\lambda = \bar{\lambda}$, and to the extent that consumption is equated to the marginal utility of wealth, this introduces a strong incentive for "consumption smoothing."

The assumption that the after-tax rate of time preference in the small economy equals the given world rate of interest is a standard one in general equilibrium models of a small open economy based on intertemporal optimization; see Turnovsky (1997a). While it is strong and has been the source of criticism of the representative agent model as applied to the small open economy, this assumption is required if an interior equilibrium is to be attained, when $\rho, r,$ and τ_b, are all given constants. One justification is that a small open economy, facing a perfect world capital market, must constrain its rate of time preference by the investment opportunities available to it, and that these are ultimately determined by the exogenously given rate of return in the world capital market. For if that were not the case, the domestic agent would end up either in infinite debt or in infinite credit to the rest of the world and that would not represent a viable interior equilibrium. The economy would cease to be small.

How acceptable this assumption is depends in part upon the specific shock one is analyzing. For demand and productivity shocks, which typically leave ρ and r both unchanged, it is adequate. However, it is less satisfactory if one wishes to analyze independent changes in say ρ or r, which would break the assumed equality between them. In this case, one alternative has been to allow the rate of time preferences to be variable. This approach was first adopted by Obstfeld (1981), where he endogenizes the consumer rate of time preference through the introduction of Uzawa (1968) preferences, though this too is subject to its own criticisms.[9] An alternative approach to breaking this equality is to endogenize the world interest rate, an approach that we shall discuss in Section 6.

3.1 Equilibrium Dynamics

The complete macroeconomic equilibrium of the stationary Ramsey model can now be easily characterized. First, the consumer optimality conditions (2a) and (2b), with $\lambda = \bar{\lambda}$, together with the production function (1a'), may be solved for consumption and leisure as follows:[10]

$$C = C(\bar{\lambda}, K) \quad C_{\bar{\lambda}} < 0, C_K < 0 \tag{8a}$$

$$l = l(\bar{\lambda}, K) \quad l_{\bar{\lambda}} < 0, l_K < 0. \tag{8b}$$

An increase in the marginal utility of wealth, $\bar{\lambda}$, (which is constant and is determined by the steady state) shifts the consumption-leisure tradeoff against consumption and in favor of labor. An increase in K raises the real wage, thereby leading to a substitution of work for consumption. It is evident from (2a) that the dependence of consumption upon capital, and therefore its time dependence, arises because of (i) the interdependence between consumption and leisure in utility, and (ii) the assumption that employment is variable. If instead, employment is fixed, then consumption depends only upon $\bar{\lambda}$ and therefore also remains constant over time; there is pure "consumption smoothing."

The evolution of the system is determined by substituting the short-run equilibrium into the dynamic equations and ensuring that the transversality conditions are met. It is readily apparent that in fact the dynamics can be determined sequentially. Equations (3') and (4b) can be reduced to a pair of autonomous differential equations in the capital stock k and its shadow value q and these constitute the core of the dynamics that determine the path of real activity in the economy. Having determined the core dynamics, equation (6) then yields the dynamics of the net credit of the domestic economy. This equation may in turn be reduced to an autonomous differential equation in B, after substituting the solutions for q and K.

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[9] See e.g. Blanchard and Fischer (1989).

[10] The formal expressions for the partial derivatives of these functions are obtained by taking the differentials of (2a), (2b), and (1c').

Consider first equations (3') and (4b) rewritten as:

$$\dot{K} = I(q)K \tag{9a}$$

$$\dot{q} = r(1-\tau_b)q - \frac{(q-1)^2}{2h} - (1-\tau_y)F_K[K, 1-l(\bar{\lambda}, K)] \tag{9b}$$

where $F_K[.]$ denotes the marginal physical product of capital. Linearizing this pair of equations about the steady state $(\tilde{K}, \tilde{q})$, implies that the long-run equilibrium is a saddlepoint with stable eigenvalue $\mu < 0$. While the capital stock always evolves continuously, the shadow price of capital q may jump instantaneously in response to new information; that is, K_0 is predetermined, while $q(0)$ is freely determined. Starting from an initial capital stock, K_0, the stable dynamic time paths followed by K and q are therefore:

$$K = \tilde{K} + (K_0 - \tilde{K})e^{\mu t} \tag{10a}$$

$$q = \tilde{q} + \frac{\mu}{h}(K - \tilde{K}) \tag{10b}$$

The convexity of the adjustment cost function is an important determinant of the dynamics. In the absence of such costs, q adjusts instantaneously to its steady state equilibrium value, $\tilde{q}$, (shown below to equal unity). Capital adjusts immediately to its steady state level, with no new investment. This instantaneous adjustment is possible because the small economy facing a perfect world capital market and with no adjustment costs can purchase as much capital as it desires from the world market. It is unconstrained by its own productive capacities, as would be the case for a closed economy.

To complete the discussion of the dynamics, we must consider the national budget constraint (6). Solving this, together with the transversality condition (4c), yields the intertemporal national budget constraint:

$$B_0 + \int_0^\infty \left(\left[Y - C - I\left[1 + (h/2)I\right]\right]\right)e^{-r(1-\tau_b)t}dt = 0 \tag{11}$$

where B_0 is the initial stock of foreign bonds held by the domestic economy. If the country begins as a net creditor to the rest of the world ($B_0 > 0$), it cannot run a trade surplus indefinitely; at some point it must run a trade deficit, order for (11) to be met. The opposite holds true for a debtor nation.

This intertemporal constraint imposes an additional constraint on the evolution of the economy, determining the stable adjustment of the current account. Substituting for C and L from (8a) and (8b) linearizing, and invoking the transversality condition, one can show that

$$B_0 = \tilde{B} + \frac{\Gamma}{\mu - r(1-\tau_b)}(K_0 - \tilde{K}) \tag{12}$$

so that the solution for $B(t)$ consistent with long-run solvency is

$$B(t) = \tilde{B} + \frac{\Gamma}{\mu - r(1-\tau_b)}(K_0 - \tilde{K})e^{\mu t} \tag{13}$$

where

$$\Gamma \equiv F_K - F_L l_K - C_K - \mu > 0 .$$

Equation (12) is a linear approximation to the national intertemporal budget constraint (11) that corresponds to the stable path followed by capital, while (13) describes the relationship between the accumulation of capital and the accumulation of traded bonds during the transition.[11]

3.2 Steady State

The steady state of the economy is obtained when $\dot{K} = \dot{q} = \dot{B} = 0$ and is given by (1a'), (8a), (8b), together with

$$\tilde{q} = 1 \tag{14a}$$

$$r(1-\tau_b) = (1-\tau_y)F_K[K, 1 - l(\bar{\lambda}, K)] \tag{14b}$$

[11] This is discussed in greater detail by Turnovsky (1997a). The definition of Γ given in (13) emphasizes that capital exercises two channels of influence on the country's trade balance. First, an increase in K raises output both directly and indirectly by raising the real wage and inducing the agent to substitute labor for consumption. In addition, an increase in k lowers the shadow value of capital, q, thereby reducing investment and the economy's rate of absorption. Both of these effects improve the country's balance of trade.

$$Y = rB + C \tag{14c}$$

$$B_0 = \tilde{B} + \frac{\Gamma}{\mu - r(1-\tau_b)}(K_0 - \tilde{K}). \tag{14d}$$

These equations jointly determine the steady-state equilibrium solutions for $\tilde{C}, \tilde{l}, \tilde{K}, \tilde{Y}, \tilde{q}, \tilde{B}$, and $\bar{\lambda}$.

Several aspects of this steady state merit comment. First, the steady state value of *q* is unity, consistent with the Tobin *q* theory of investment. Second, the steady-state tax-adjusted marginal physical product of capital is equated to the exogenously given tax-adjusted foreign interest rate, thereby determining the domestic capital-labor ratio in precisely the same ways as it is determined by the rate of time discount in a closed economy. Third, (14c) implies that in steady-state equilibrium, the current account balance must be zero. Equation (14d) describes the equilibrium relationship between the accumulation of capital over time and the accumulation of traded bonds consistent with the nation's intertemporal budget constraint. Through this relationship the steady-state depends upon the initial stocks of assets K_0, B_0, as a result of which a *temporary* policy (or other shocks) has a *permanent* effect. This is an important policy implication of the knife-edge condition (7) and the zero root that it introduces into the dynamics. Finally, we should note that this steady state is sustainable only as long as the government maintains a feasible debt and taxing policy consistent with its intertemporal budget constraint.[12]

4. The Endogenous Growth Model

The investment-based endogenous growth model has recently been a subject of intensive research. Most such models are based on the assumption that the supply of labor is inelastic (an important knife-edge assumption) and, as we shall demonstrate, the endogeneity or otherwise of labor is important insofar as the determination of the equilibrium growth rate is concerned.

[12]The assumption of a continuously balanced budget ensures that the government's static and intertemporal budget constraint is automatically met. The case of debt financing and the constraints imposed by intertemporal solvency is discussed by Turnovsky (1997a).

The key feature of the endogenous growth model is that it is capable of generating ongoing growth in the absence of population growth. For this to occur, the production function (1a) must have constant returns to scale in the accumulating factors, individual and aggregate capital, that is,

$$\sigma + \eta = 1 \tag{15}$$

Substituting this into (1a), this implies individual and aggregate production functions of the form

$$Y_i = \alpha\left[(1-l)K\right]^{\eta} K_i^{\sigma}; \quad Y = \alpha\left[(1-l)N\right]^{\eta} K \tag{16}$$

The individual production function is thus constant returns to scale in private capital, K_i, and in labor, measured in terms of "efficiency units" $(1-l)K$. Summing over agents, the aggregate production function is thus linear in the accumulating capital stock. Note further, that as long as $\eta \neq 0$ so that there is an aggregate externality, the average (and marginal) productivity of capital depends upon the size of the population. Increasing the population, holding other technological characteristics constant, increases the productitivy of capital and the equilibrium growth rate. The economy is thus said to have a "scale effect"; see Jones (1995a). Such scale effects run counter to the empirical evidence and have been a source of criticism of the AK growth model; see Backus, Kehoe and Kehoe (1992). These scale effects can be eliminated from the AK model if either (i) there are no externalities ($\eta = 0$), or (ii) if the individual production function (1a) is modified to

$$Y_i = \alpha(1-l)^{1-\sigma} K_i^{\sigma}\left(K/N\right)^{\eta}$$

so that the externality depends upon the average, rather than the aggregate capital stock; see Mulligan and Sala-i-Martin (1993). Henceforth throughout this section, we shall normalize the size of the population at $N = 1$ and thereby eliminate the issue of scale effects.

4.1 Inelastic Labor Supply

We begin with the case where labor is supplied inelastically, i.e. $l = \bar{l}$. With population normalized, the individual and aggregate production functions are of the pure AK form:

$$Y_i = AK_i; \quad Y = AK \tag{17}$$

where $A \equiv \alpha(1-\bar{l})^{1-\sigma}$ is a fixed constant. With the labor supply fixed, both the marginal and average productivity of capital are constant. The specification of the technology, consistent with ongoing growth, is a very strong knife-edge condition, one for which the endogenous growth model has been criticized; see Solow (1994).[13]

To determine the macroeconomic equilibrium, we first take the time differential of (2a), and then combine the resulting equation with (4a), which, with fixed employment (and normalized population) implies

$$\frac{\dot{C}}{C} = \frac{r(1-\tau_b)-\rho}{1-\gamma} \equiv \psi \tag{18}$$

Because we are in a context of ongoing growth, we do not need to impose the knife-edge condition (7) introduced to ensure a finite steady state consumption level in the stationary Ramsey model. Instead, an immediate consequence of (18) is that the equilibrium growth rate of domestic consumption is proportional to the difference between the after-tax return on foreign bonds and the (domestic) rate of time preference. From a policy perspective, it also implies that the consumption growth rate varies inversely with the tax on foreign bond income, but is independent of all other tax rates. Solving this equation implies that the level of consumption at time *t* is

$$C(t) = C(0)e^{\psi t} \tag{19}$$

where the initial level of consumption $C(0)$ is yet to be determined.

The critical determinant of the growth rate of capital is the relative price of installed capital, *q*, the path of which is determined by the arbitrage condition (4b). To analyze this further, we rewrite (4b) as the following nonlinear differential equation with constant coefficients:

$$\dot{q} = 2h[r(1-\tau_b)q - A(1-\tau_y)] - (q-1)^2 \tag{20}$$

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[13] Note that the technology (18) is identical to that of the original Harrod-Domar model, to which the AK model is a modern counterpart. In fact it was Harrod,himself, who referred to the "knife-edge" characteristics of his model.

In order for the capital stock domiciled in the economy ultimately to follow a path of steady growth (or decline), the stationary solution to this equation attained when $\dot{q}=0$, must have (at least) one real solution. Setting $\dot{q}=0$ in (20), implies that the steady-state value of q, $\tilde{q}$ say, must be a solution to the quadratic equation:

$$A(1-\tau_y)+\frac{(q-1)^2}{2h}=rq(1-\tau_b) \tag{21}$$

A necessary and sufficient condition for the capital stock ultimately to converge to a steady growth path is that this equation have real roots, and this will be so if and only if

$$A(1-\tau_y)\le r(1-\tau_b)\left[1+\frac{hr(1-\tau_b)}{2}\right] \tag{22}$$

There is a tradeoff between the size of the adjustment cost, h, and the productivity of capital, A, consistent with the existence of a balanced growth path for capital. If for a given h, A is sufficiently large to violate (22), the returns to capital dominate the returns to bonds, irrespective of the price of capital, so that no long-run balanced equilibrium exists along which the returns on the two assets are equalized.

Suppose that (21) has real roots which we denote by q_1 (smaller) and q_2 (larger), respectively. It is easily shown that the equilibrium point that corresponds to the smaller equilibrium value, q_1, is unstable, while the equilibrium that corresponds to the larger value, q_2, although locally stable, violates the transversality condition (4c). The behavior of equilibrium q can thus be summarized by:

Proposition 1: The only solution for q which is consistent with the transversality condition is that q always be at the (unstable) steady-state solution q_1, given by the negative root to (21). *Consequently there are no transitional dynamics in the market price of capital q.* In response to any shock, q immediately jumps to its new equilibrium value.

Correspondingly, domestically domiciled capital is always on its steady growth path, growing at the rate $\varphi = (q_1 - 1)/h$.

The domestic government is assumed to maintain a continuously balanced budget, in accordance with (5), while the domestic current account is given by (6). Substituting the expressions for I and $K(t)$ from (3'), and $C(t)$ from (20) into (6), and invoking the transversality conditions, we can determine (i) $C(0)$ consistent with the nation being internationally solvent , and (ii) the resulting time path for traded bonds; see Turnovsky (1996). This equation, together with (3') and (19), the solution for *q*, and the initial condition on $C(0)$, comprise a closed form solution describing the evolution of the small open economy starting from given initial stocks of traded bonds B_0 and capital stock K_0. The following general characteristics of this equilibrium can be observed.

(i) Consumption on the one hand, and physical capital on the other, are always on their steady-state growth paths, growing at the rates ψ and φ respectively. The former is driven by the difference between the after-tax rate of return on foreign bonds and the domestic rate of time preference. The latter by *q*, which is determined by the technological conditions in the domestic economy, as represented by the (fixed) marginal physical product of capital *A*, and adjustment costs *h*, relative to the return on foreign assets. For the simple linear production function, the rate of growth of capital also determines the equilibrium growth of domestic output $\dot{Y}/Y$.

Thus an important feature of this equilibrium is that it can sustain differential growth rates of consumption and domestic output. This is a consequence of the economy being small and open. It contrasts to the closed economy in which, constrained by the growth of its own resources, all real variables, including consumption and output, would ultimately have to grow at the same rate. But it is also a consequence of the knife-edge assumption being made and ceases to hold in other models.[14]

(ii) Holdings of traded bonds are subject to transitional dynamics, in the sense that their growth rate $\dot{B}/B$ varies through time. Asymptotically the growth rate converges to $\max[\psi, \varphi]$ and

[14] We assume that the country is sufficiently small so that it can maintain a growth rate which is unrelated to that in the rest of the world. Ultimately, this requirement imposes a constraint on the growth rate of the domestic economy. If it grows faster than does the rest of the world, at some point it will cease to be small. While we do not attempt to resolve this issue here, we should note that the question of convergence of international growth rates is an area of intensive research activity; see Mankiw, Romer, and Weil (1992), Barro and Sala-i-Martin (1992a), Quah, (1996), Galor (1996).

which it will be depends critically upon the size of the consumer rate of time preference relative to the rates of return on investment opportunities; see Turnovsky (1996).

(iii) With all taxes being fully rebated and labor supply fixed, the consumption tax is completely neutral. It has no effect on any aspect of the economic performance and acts like a pure lump-sum tax.

Income Taxes and Growth

Differentiating the solution for q, (see (21)), together with the definition of the growth rate of capital φ, and consumption ψ, the effects of the two income taxes can be summarized by:

> **Proposition 2:** An increase in the tax on bond income increases the growth rate of capital and reduces the growth rate of consumption; an increase in the tax on domestic income reduces the growth rate of capital, but leaves the growth rate of consumption unaffected.

Intuitively, a higher tax on bond income lowers the rate of return on bonds, thereby inducing investors to increase the proportion of capital in their portfolios, raising the price of capital and inducing growth in capital. In addition, this tax induces agents to switch from saving to consumption, increasing the ratio of consumption to wealth. This slows down the rate of growth of consumption. A higher tax on domestic income generates the opposite portfolio response, lowering the growth rate of capital.

4.2 Elastic Labor Supply

The endogenous growth model we have been discussing includes two interdependent criticial knife-edge restrictions: (i) inelastic labor supply, and (ii) fixed productivity of capital. The structure of the equilibrium changes fundamentally when the labor supply is endogenized. This introduces two key changes. The first is that the production function is modified to (16), so that the productivity of capital now depends positively upon the fraction of time devoted to labor. Second, the fixed endowment of a unit of time leads to the requirement that the steady-state allocation of time between labor and leisure must be constant.

To derive the macroeconomic equilibrium, we begin by dividing the optimality condition (2a) by (2b), the marginal rate of substitution condition between consumption and leisure, to obtain:

$$\frac{C_i}{Y_i} = \frac{C}{Y} = \left(\frac{l}{1-l}\right)\frac{1-\sigma}{\theta} \tag{23}$$

Next, taking the time derivatives of: (i) the optimality condition for consumption, (2a), (ii) the equilibrium consumption-output ratio, (23), and, (iii) the production function, (7), leads to:

$$(\gamma-1)\frac{\dot{C}}{C} + \theta\gamma\frac{\dot{l}}{l} = \frac{\dot{\lambda}}{\lambda} = \rho - r(1-\tau_b) \tag{24a}$$

$$\frac{\dot{C}}{C} - \frac{\dot{Y}}{Y} = \frac{\dot{l}}{l} + \frac{\dot{l}}{1-l} \tag{24b}$$

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} - (1-\sigma)\frac{\dot{l}}{1-l} = \frac{q-1}{h} - (1-\sigma)\frac{\dot{l}}{1-l} \tag{24c}$$

Combining these equations with (16), (3'), and (4b), the macroeconomic equilibrium can be expressed by the pair of differential equations in q and l:

$$\dot{q} = r(1-\tau_b)q - \frac{(q-1)^2}{2h} - (1-\tau_y)\alpha(1-l)^{1-\sigma} \tag{25a}$$

$$\dot{l} = \frac{1}{F(l)}\left[r(1-\tau_b) - \rho - \frac{(1-\gamma)(q-1)}{h}\right] \tag{25b}$$

where $F(l) \equiv \frac{(1-\gamma)\sigma}{1-l} + \frac{1-\gamma(1+\theta)}{l} > 0$.

The steady state to (25) is obtained by setting $\dot{q} = \dot{l} = 0$ and is therefore characterized by the relative price of capital, q, and the fraction of time devoted to leisure, l, both being constant. Linearizing (25) around its steady state, we can easily show that the two eigenvalues to the linearized approximation are both positive; see Turnovsky (1999). Hence the only bounded equilibrium is one in which both q and l adjust instantaneously to ensure that the economy is always on its balanced growth path (denoted by ~) namely:[15]

$$\tilde{\psi} = \frac{\tilde{q}-1}{h} = \frac{r(1-\tau_b)-\rho}{1-\gamma} \tag{26a}$$

$$\frac{(1-\tau_y)\alpha(1-\tilde{l})^{1-\sigma}}{\tilde{q}} + \frac{(\tilde{q}-1)^2}{2h\tilde{q}} = r(1-\tau_b) \tag{26b}$$

where the transversality condition now implies:

$$\tilde{\psi} = \frac{\tilde{q}-1}{h} < r(1-\tau_b)$$

Equation (26a) implies that the equilibrium is one in which domestic output, capital, and consumption all grow at a common rate determined by the difference between the world rate of interest and the domestic rate of time preference, all multiplied by the intertemporal elasticity of substitution. The form of the expression is analogous to the equilibrium growth rate in the simplest AK model; see Barro (1990). The only difference is that for the small open economy the (fixed) marginal physical product of capital is replaced by the (given) foreign interest rate. Given this growth rate, (26a) determines the equilibrium price of capital, $\tilde{q}$, which will ensure that domestic capital grows at this equilibrium rate. Having obtained $\tilde{q}$, (26b) then determines the fraction of time devoted to leisure (employment) such that the marginal physical product of capital ensures that the rate of return on domestic capital equals the (given) world rate of interest. Hence in this small open economy with elastically supplied labor, the growth rate of output and capital is independent of production characteristics such as the productivity parameter, α, and the marginal cost of adjustment, h. Changes in these parameters are reflected in the labor-leisure choice $\tilde{l}$. In order for

[15] This local instability of the dynamic path depends in part upon our assumptions of a Cobb-Douglas production function and constant elasticity utility function, and justifies our focus on that equilibrium in the present analysis. For more general production functions one cannot dismiss the possibility that the dynamics has a stable eigenvalue, giving rise to potential problems of indeterminate equilibria. In a model with both physical and nonhuman capital, Benhabib and Perli (1994), Ladrón-de-Guevara, Ortigueira, and Santos (1997) show how the steady-state equilibrium may become indeterminate. Other authors have emphasized the existence of externalities as sources of indeterminacies of equilibrium; see Benhabib and Farmer (1994).

the equilibrium to be viable, the implied fraction of time devoted to leisure must satisfy $0 < l < 1$. This will be so if and only if:

$$0 < r(1-\tau_b) + \left(\frac{h}{1-\gamma}\right)\left(r(1-\tau_b) - \rho\right)\left[\frac{r(1-\tau_b)(1-2\gamma)+\rho}{2(1-\gamma)}\right] < (1-\tau_y)\sigma\alpha \tag{27}$$

a condition that is plausibly met; see Turnovsky (1999).

The viability of the equilibrium requires that the nation's intertemporal budget constraint, which in this growth context is

$$B_0 + \frac{K_0}{r-\psi}\left[\left[1 - \left(\frac{\tilde{l}}{1-\tilde{l}}\right)\frac{(1-\sigma)}{\theta}\right]\frac{\tilde{Y}}{\tilde{K}} - \left(\frac{\tilde{q}^2-1}{2h}\right)\right] = 0 \tag{28}$$

be met. The initial value of its foreign bonds plus the capitalized value of the current account surplus along the balanced growth path must sum to zero. Having determined the equilibrium values of $\tilde{l}, \tilde{q},$ and $\widetilde{Y/K}$, the intertemporal constraint (28) determines the combination of the initial capital stock, K_0, and the initial stock of foreign bonds, B_0, necessary for the equilibrium to be intertemporally viable. Substituting (28) into the current account relationship, we see that the equilibrium stock of traded bonds accumulate at the common equilibrium growth $\tilde{\psi}$.

If the inherited stocks of these assets violate (28) we assume that the appropriate adjustment is attained through initial lump-sum taxation (if necessary), of the form $dT_0 + dB_0 + \tilde{q}dK_0 = 0$, whereby the private agent is forced to readjust his portfolio to attain the intertemporally viable ratio consistent with (29). Using the government's balanced budget condition (5) one can determine the required level of lump sum taxes at each point in time. Along the balanced growth path, this is of the form:

$$T(t) = (aK_0 + bB_0)e^{\tilde{\psi}t} \tag{29}$$

where *a*, *b* are constants, easily derived from the balanced growth equilibrium.

Comparison with Fixed Employment AK Model

Endogenizing labor fundamentally changes the macrodynamic equilibrium from its determination in the pure AK model, described in Section 4.1, where the labor supply is fixed. The key to this is the marginal rate of substitution relationship, (23), which implies that, because in equilibrium the allocation of time must be constant, the long-run consumption-output ratio must also be constant, forcing equilibrium consumption and output to grow at the same rate. The divergence in the equilibrium growth rates between consumption and output, which with fixed labor supply could prevail, is eliminated through the adjustment in the time allocation between work and leisure.

This contrasting mechanism is reflected in the corresponding dynamics of the external sector. As noted, with fixed employment, these differential growth rates are sustained by the transitional dynamics in the accumulation of traded bonds from their arbitrarily given initial stock, B_0, with the initial consumption being determined to ensure that international solvency is met. In the present case, however, the initial consumption level is determined by the optimality condition (23). Now, however, the initial stocks of assets B_0, K_0 must be chosen (through some form of intervention) to ensure that the economy remains internationally solvent.

This seemingly modest change in economic structure has significant consequences. It implies that factors that under fixed employment are reflected in the equilibrium growth rate of output are now reflected in the allocation of work time. Thus, for example, whereas in the fixed employment AK model the growth rate of output depends upon the domestic production parameters, α and h, it is now independent of these parameters, which instead influence l. Particularly important differences arise with respect to fiscal policy. From (26a) and (26b) we can summarize their effects as follows:

> **Proposition 3:** An increase in the domestic income tax increases the time devoted to work, that is reduces leisure. An increase in the tax on bond income leads to a reduction in work and an increase in leisure. Neither of these taxes affects the growth rate of output, although a higher tax on bond income reduces the growth rate of consumption.

The intuition is as follows. With the equilibrium growth rate and the equilibrium price of capital, $\tilde{q}$, fixed, a higher domestic income tax reduces the after-tax return to capital. In order to maintain equilibrium among rates of return, the productivity of capital must be increased. This is achieved by an increase in the fraction of time devoted to labor, that is, by a decline in leisure. An increase in the tax on foreign bond income generates the opposite portfolio response. The lower return on foreign bonds requires a lower equilibrium return on domestic capital, which is accomplished by a reduction in the productivity of capital brought about by an increase in leisure.

Both these responses contrast with the fixed-employment open economy AK model, where each affects the equilibrium growth rate of output as summarized in Proposition 2. On the other hand, the consumption tax has no effect either on the growth rate or on employment. Its only effect is on the consumption-output ratio, which is reduced. In this respect, the tax acts very much like a lump-sum tax, as in the fixed employment AK model of the open economy.[16]

The fact that the growth rate is independent of most income taxes (except τ_b) offers an interesting perspective to the following issue. As we observed at the outset, one of implications of the basic endogenous growth model, a feature that distinguishes it from the traditional neoclassical model, is that its equilibrium growth rate varies inversely with distortionary income taxes. The fact that empirical evidence by Easterly and Rebelo (1993), Stokey and Rebelo (1995) and Jones (1995a, 1995b) does not support this implication, has been used as evidence against these endogenous growth models. Our results suggest some caution might be required in reaching this conclusion. For small economies facing a perfect world capital market, equilibrium growth rates are in fact independent of most tax rates. Instead, such economies respond to changes in tax rates through variations in their equilibrium labor-leisure choice.

5. Non-Scale Growth Model

[16]As a further comparison, the effects of both the domestic income tax and the consumption tax contrast sharply with the analogous closed economy with endogenous labor. In such an economy both lead to a reduction in the growth rate together with an increase in leisure; see Turnovsky (2000). With the equilibrium growth rate fixed from the consumption side, this tradeoff ceases to exist in the small open economy.

As noted, the endogenous growth model has been criticized on both empirical and theoretical grounds. We therefore now turn to the third class of model, the non-scale model, which imposes no restrictions on the technological parameters in the production function. The increased flexibility of the production function is associated with a higher order dynamics in comparison to the corresponding AK growth model. Thus, in cases where the AK model is always on its balanced growth path, the corresponding non-scale model will follow a first-order adjustment path.[17]

3.2 Inelastic Labor Supply

Our objective is to analyze the dynamics of the aggregate economy about a balanced growth path. Along such an equilibrium path, aggregate output and the aggregate capital stock are assumed to grow at the same constant rate rates, so that the aggregate output-capital ratio remains unchanged. Summing the individual production functions (1a) over the N agents, the aggregate production function with inleastic labor supply is:

$$Y = \alpha(1-\bar{l})^{1-\sigma} K^{\eta+\sigma} N^{1-\sigma} \equiv AK^{\sigma_K} N^{\sigma_N} \tag{30}$$

where A is defined below (17), $\sigma_N \equiv 1-\sigma$ = share of labor in aggregate output, $\sigma_K \equiv \eta+\sigma$ = share of capital in aggregate output. Thus $\sigma_K + \sigma_N = 1+\eta$ measures total returns to scale of the social aggregate production function. Taking percentage changes of (30) and imposing the long-run condition of a constant Y/K ratio, the long-run equilibrium growth of capital and output, g, is

$$g \equiv \left(\sigma_N/(1-\sigma_K)\right)n > 0 \tag{31}$$

Equation (31) exhibits the key feature of the non-scale growth model, namely, that the long-run equilibrium rate is proportional to the population growth rate by a factor that reflects the productivity of labor and capital in the aggregate production function.[18] In particular, it is

[17] Eicher and Turnovsky (2000) discuss the dynamic characteristics of the two-sector non-scale model in some detail. They show how the stable adjustment path is now second order, thereby introducing important flexibility into the analysis of transitional dynamics.

[18] Eicher and Turnovsky (1999a) provide a detailed characterization of the determinants of long-run equilibrium growth in a two-sector non-scale growth model.

independent of any macro policy instrument. We shall show below that as long as the dynamics of the system are stable, $\sigma_K < 1$, in which case the long-run equilibrium growth is $g > 0$, as indicated. Under constant returns to scale, $g = n$, the rate of population growth, as in the standard neoclassical growth model, to which the present one-sector model reduces. Otherwise *g* exceeds *n* or is less than *n*, that is there is positive or negative per capita growth, according to whether returns to scale are increasing or decreasing, $\eta \gtrless 0$.

To analyze the transitional dynamics of the economy about the long-run stationary growth path, it is convenient to express the system in terms of the relative price of installed capital, *q*, and the following stationary variables:

$$c \equiv C/N^{(\sigma_N/(1-\sigma_K))}; \quad k \equiv K/N^{(\sigma_N/(1-\sigma_K))}; \quad b \equiv B/N^{(\sigma_N/(1-\sigma_K))} \tag{32}$$

Under standard conditions of constant social returns to scale $[\sigma_N + \sigma_K = 1]$ and the quantities in (32) reduce to standard per capita quantities; i.e. $c = C/N = C_i$, etc. Otherwise they represent "scale-adjusted" per capita quantities.

Consumption Dynamics

To determine the growth rate of consumption we take the time derivative of (2a) and combine with (4a) to find that the individual's consumption grows at the constant rate:

$$\dot{C}_i/C_i = \big((r(1-\tau_b) - \rho - n)/(1-\gamma)\big) \equiv \psi_i \tag{33}$$

With all individuals being identical, the growth rate of aggregate consumption is $\psi = \psi_i + n$, so that

$$\dot{C}/C = \big((r(1-\tau_b) - \rho - \gamma n)/(1-\gamma)\big) \equiv \psi \tag{33'}$$

Differentiating *c* in (32) and using (33'), the growth rate of the scale-adjusted per capita consumption is:

$$\dot{c}/c = \big((r(1-\tau_b) - \rho - \gamma n)/(1-\gamma)\big) - \big(\sigma_N/(1-\sigma_K)\big)n \equiv \psi - g \tag{34a}$$

Equations (33), (33') and (34a) all share the property that with a perfect world capital market, the corresponding consumption growth rates are constant and independent of the production characteristics of the domestic economy. In addition, these equilibrium growth rates vary inversely with the tax on foreign bond income, but are independent of all other tax rates. These aspects of the dynamics of consumption remain unchanged from the AK model discussed in Section 4.1

Capital and the Price of Capital

The dynamics of capital accumulation are, however, distinctly different from those of the AK model with fixed labor supply, in which, like consumption, capital always lies on its balanced growth path. In the present model we find that the scale-adjusted capital-labor ratio, *k*, and the relative price of capital, *q*, converge to a long-run steady growth path along a transitional locus. To derive this path we differentiate *k* in (32) with respect to time and combine with (3'), to obtain:

$$\dot{k}/k = \left[\left((q-1)/h\right) - \left(\sigma_N/(1-\sigma_K)\right)n\right] = \varphi - g \qquad (34b)$$

To derive the law of motion for the relative price of the capital good, we substitute the production function, (1a), the aggregation condition, $K = NK_i$, and (32) into the arbitrage condition (4b). The latter can then be expressed as

$$\dot{q} = r(1-\tau_b)q - (q-1)^2/2h - (1-\tau_y)A\sigma k^{\sigma_K - 1} \qquad (34c)$$

Thus (34b) and (34c) comprise a pair of equations in *q* and *k*, that evolve independently of consumption.

In order for the domestic capital stock ultimately to follow a path of steady growth, the stationary solution to (34b), (34c), attained when $\dot{q} = \dot{k} = 0$, must have (at least) one *real* solution. Setting $\dot{q} = \dot{k} = 0$ we see that the steady-state values of *q* and *k*, $\tilde{q}$ and $\tilde{k}$, are determined recursively as follows. First, the steady-state price of installed capital is:

$$\tilde{q} = 1 + h\left(\sigma_N/(1-\sigma_K)\right)n = 1 + hg \qquad (35a)$$

Having determined $\tilde{q}$ from this equation, the equilibrium scale-adjusted capital-labor ratio, $\tilde{k}$, is then determined from the steady-state arbitrage condition:

$$(1-\tau_y)A\sigma\tilde{k}^{\sigma_K -1} + (\tilde{q}-1)^2/2h = r(1-\tau_b)\tilde{q} \qquad (35b)$$

To be viable, the long-run equilibrium must satisfy the transversality conditions. Substituting (4a), (3') into (4c) and evaluating this requires that:

$$r(1-\tau_b) > g > 0\,, \qquad (36)$$

That is, the after-tax interest rate on foreign bonds must exceed the growth rate of domestic aggregate output. Observe that the condition (36) ensures that (35a) and (35b) imply a unique steady state equilibrium having: (i) a positive equilibrium growth rate of capital (output), $\tilde{\varphi} = g$, and, (ii) a positive scale-adjusted capital-labor ratio, $\tilde{k}$.[19]

The equilibrium of the production side is thus fundamentally different in the non-scale economy from that of the simple AK technology. First, it is characterized by transitional dynamics, (34b), (34c), the nature of which will be discussed below. Second, in contrast to the AK technology where the existence of a balanced equilibrium growth rate depends upon the size of the adjustment costs relative to the productivity of capital, as in (22), the transversality condition (4a) *always* ensures the existence of a unique equilibrium growth rate of capital.[20] Moreover, as is evident from (31), the steady-state growth rate of aggregate capital (and output) shares the standard characteristic of non-scale growth models, namely that it is: (i) strictly positive and (ii) depends upon the returns to scale in the production function. Specifically, the growth rate is greater than or less than that of labor, according to whether there are increasing or decreasing returns to scale in aggregate production. Furthermore, it is independent of the taxes levied upon interest or capital income. This implication contrasts with the AK model of Section 4.1 in which the growth of output increases with

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[19]This may be shown as follows:

$$(1-\tau_y)\alpha\sigma\tilde{k}^{\sigma_K -1} = r(1-\tau_b)\tilde{q} - \left(\tilde{q}-1\right)^2/2h$$

Using (10a) and (11) the right hand of this equation exceeds $g(1+hg/2) > 0$, thus implying $\tilde{k}^{\sigma_K -1} > 0$ and hence $\tilde{k} > 0$.

[20]The AK model corresponds to $\sigma + \eta = \sigma_K = 1$ in (10b). This yields a quadratic equation in q, which may or may not have a real solution. In the case that it does, the smaller root yields the equilibrium growth rate of output $\varphi = (q-1)/h$.

the former tax rate and decreases with the latter. In the non-scale model, the response to taxes occurs through the adjustment in *k*.

One further contrast is the response of the relative price of capital, *q*, to the cost of adjustment *h*. From (21), we see that in the AK model an increase in *h* lowers the return to capital arising from its favorable impact on installation costs; see (4b). For the return to capital to remain equal to the fixed return on foreign bonds, a *q* must *decline*. In the non-scale economy, with the equilibrium growth rate determined by production elasticities, an increase in *h* requires a *higher q*, in order for the growth rate of capital to equal the equilibrium growth rate of output; see (35a).

Linearizing (35b) and (35c) around (36a) and (36b), the local transitional dynamics of capital and its shadow price can be represented by the system:

$$\begin{pmatrix}\dot{k}\\ \dot{q}\end{pmatrix}=\begin{pmatrix}0 & \tilde{k}\big/h\\ -(1-\tau_y)A\sigma\left(\sigma_K-1\right)\tilde{k}^{\sigma_K-1} & r(1-\tau_b)-g\end{pmatrix}\begin{pmatrix}k-\tilde{k}\\ q-\tilde{q}\end{pmatrix} \qquad (37)$$

It is immediately apparent from (37) that $\sigma_K < 1$ is a necessary and sufficient condition for saddlepoint , stability, in which case (35a) ensures that the equilibrium growth rate of output, *g* is positive. The stability condition asserts $\eta < \sigma_N = 1-\sigma$, so that the share of external spillover generated by private capital accumulation, and hence the overall social increasing returns to scale, cannot exceed the exogenously growing factor's share (labor) in production As usual, we assume that the capital stock accumulates slowly, so that *k* evolves gradually from its initial value, k_0, while the shadow value of capital may adjust instantaneously to new information.

As in the AK model with inelastic labor supply, an important aspect of this equilibrium is that differential growth rates of consumption and domestic output can be sustained. This is a consequence of the economy being small in the world bond market and as in that model is associated with transitional dynamics in foreign asset accumulation.

5.2 Elastic Labor Supply

The knife-edge relationship in the above version of the non-scale model is the inelastic labor supply. Once this is imposed, all other parameters are unrestricted. However, when this assumption is relaxed and labor is endogenously supplied, a much stronger knife-edge condition is required. This is because the optimality condition (23) must now be taken into account. With the fraction of time allocated to work constant in steady state, this relationship implies that the steady-state consumption-output ratio must be constant, thereby imposing the equality of the long-run growth rates of consumption and output (and capital). Thus equating (34') to (32) we must have:

$$\psi \equiv \frac{r(1-\tau_b)-\rho-\gamma n}{1-\gamma} = \left(\frac{\sigma_N}{1-\sigma_K}\right)n \equiv g \qquad (38)$$

That is, the return on foreign bonds, given the taste parameters, must be such that the implied growth rate of consumption is driven to that of capital, which is determined by the population growth rate in conjunction with the productive elasticities, in accordance with the non-scale growth model,

The condition (38) is the growth analogue to the knife-edge condition (7) in the stationary Ramsey model, to which it reduces in the absence of growth (n=0). Note, further, that if there are constant returns to scale, (38) simplifies to

$$r(1-\tau_b) = \rho + n \qquad (38')$$

which is the familiar long-run viability condition for the standard Ramsey growth model. But now, with the more general productive structure, (38) both the productive elasticities, σ_K, σ_N as well as the intertemporal elasticity measure, γ.

Macrodynamic Equilibrium

Being a generalization of the previous models, the macrodynamic equilibrium of the present model includes elements of the earlier discussion. Here, we merely sketch its structure, thereby enabling us to identify the role played by the more general knife-edge condition (38). Following the procedures of Sections 4 and 5.1, the macrodynamic equilibrium includes the following set of equations:

$$\dot{k} = \left(\frac{q-1}{h} - g\right)k \tag{34b}$$

$$\dot{q} = r(1-\tau_b)q - \frac{(q-1)^2}{2h} - (1-\tau_y)\alpha(1-l)^{\sigma_N}\sigma k^{\sigma_K - 1} \tag{34c'}$$

$$\dot{l} = \frac{(1-\gamma)\sigma_K}{F(l)}\left(g - \frac{q-1}{h}\right) \tag{25b'}$$

where in deriving (25') we have used the condition (38). Note that (34b) and (25') imply that scale-adjusted capital, *k*, and leisure, *l*, move in inverse proportion,

$$\dot{l}(t) = -\frac{(1-\gamma)\sigma_K}{F(\tilde{l})k}\dot{k}(t) \tag{39}$$

and to a linear approximation, the distances from their respective steady states (denoted by tildes) are related by:

$$l(t) - \tilde{l} = -\frac{(1-\gamma)\sigma_K}{F(\tilde{l})\tilde{k}}\left(k(t) - \tilde{k}\right) \tag{39'}$$

Intuitively, as capital increases, the return to labor rises and the desirability of leisure declines. Note from (39) that employment is subject to transitional dynamics that mirrors the path of capital.

Thus (39) introduces a linear dependence into the three dynamic equations, thus implying that the stationary equations corresponding to (34b), (34c') and (25b') do not suffice to determine the steady state. Setting $\dot{k} = \dot{l} = 0$, we see that both (34b) and (25b') imply that the steady-state price of installed capital, $\tilde{q}$, remains as determined by (36a). Given this value of $\tilde{q}$, the remaining steady-state relationship [obtained by setting $\dot{q} = 0$ in (34c')] determines only the equilibrium marginal physical product of capital, which except in polar cases depends upon *both* $\tilde{l}$ and $\tilde{k}$. If employment is fixed (as in Section 5.1), then this determines $\tilde{k}$; if $\sigma_K = 1$, so that we have an AK technology (as in Section 4.2), then this determines $\tilde{l}$.

But in the present case, where both $\tilde{l}$ and $\tilde{k}$ are endogenously determined, further consideration is required to pin each down. The additional relationship is the current account. This requires that $\tilde{l}$ be appropriately chosen to ensure that (23) generates a consumption path that is consistent with the nation's intertemporal budget constraint. The argument is basically similar to that of the stationary Ramsey model discussed in Section 3. It proceeds as follows. Linearizing (34b) and (34c') and substituting (39'), one can show that the stable transitional adjustment path for *k* and *q* are given by:

$$k(t) = \tilde{k} + (k_0 - \tilde{k})e^{\mu t} \tag{40a}$$

$$q(t) = \tilde{q} + \frac{h\mu}{\tilde{k}}(k(t) - \tilde{k}) \tag{40b}$$

where the stable eigenvalue $\mu < 0$. Now take the nation's current account relationship (6) and express it in terms of the scale-adjusted quantity *b*. Noting (23) this becomes

$$\dot{b} = (r - g)b + \alpha(1-l)^{1-\sigma} k^{\sigma_K}\left(1 - \frac{1-\sigma}{\theta}\frac{l}{1-l}\right) - \frac{(q^2-1)}{2h}k \equiv (r-g)b + H(k,l,q) \tag{41}$$

Linearizing this equation and solving, using (39), (40a), (40b), and the transversality condition, (41) leads to a constraint of the form:

$$G(\tilde{k}, \tilde{l}, \tilde{q}, k_0, b_0) = 0 \tag{42}$$

This is the analogue to (12) in the stationary Ramsey model. Given $\tilde{q}$ from (35a), (42) is an intertemporal solvency condition, which in conjunction with the stationary condition for (34c'), determines the equilibrium scale-adjusted capital stock, $\tilde{k}$, and the fraction of time devoted to leisure, $\tilde{l}$, as functions of the initial stocks of assets, k_0, b_0. The macroeconomic equilibrium is thus fully determined, and indeed its structure now closely parallels that of the stationary Ramsey model.

6. Imperfect World Financial Market

One knife-edge condition that plays a central role throughout this paper is the one that arises from the arbitrage condition, (4a), together with the assumption that both the rate of time preference and the world interest rate are taken as given constants. One way to break this tight relationship is to endogenize either of these parameters. We have already noted that one option is to endogenize the rate of time preference, ρ, but this is associated with unattractive characteristics. To derive well-behaved dynamics, requires the implausible assumption that the rate of time preference (impatience) increases with wealth.

An alternative, and more appealing way to break the knife-edge is to drop the assumption that the economy can borrow or lend as much as it wants at a fixed interest rate. Indeed, it is realistic to argue that economies face limitations in their access to the world financial markets. This can be most conveniently addresses in the case of a debtor nation by postulating that the rate of interest at which it may borrow is say an increasing function of its debt. This type of constraint was originally proposed by Bardhan (1967) and has been introduced by many authors since then; see Obstfeld (1982), Bandari, Haque, and Turnovsky (1990), Fisher (1995). While these specifications are essentially ad hoc, more formal justification of this type of relationship, in terms of default risk, has been provided by Eaton and Gersovitz (1981, 1989) and others.

One issue that arises is whether the specification of debt cost is expressed in terms of the absolute level of debt, as originally proposed by Bardhan, or relative to some earning capacity to service the debt. The latter measure is appropriate if an equilibrium of ongoing growth is to be sustained. The critical point is that endogenizing the interest rate relaxes the knife-edge condition, fundamentally changing the structure of the steady-state equilibrium.

To illustrate the impact of endogenizing the world interest rate, let us focus on a borrowing economy and assume that the cost of foreign borrowing is $r(B/K)$, where B now refers to debt (rather than credit). We assume that the costs of foreign debt depend upon the economy's aggregate indebtedness, that the individual agent takes as given. In the case of the stationary Ramsey model, the critical adjustment is to the arbitrage condition (4a), which now becomes:

$$\rho - (\dot{\lambda}/\lambda) = r(B/K)(1-\tau_b) \tag{4a'}$$

The stationary solution to this is

$$r(B/K)(1-\tau_b) = \rho \tag{7'}$$

With the foreign borrowing costs being endogenous, this condition no longer imposes a constraint on the parameters. Rather it determines the economy's steady-state level of indebtedness B/K, so as to drive the equilibrium cost of borrowing (or return on lending) to the exogenously given rate of time preference. Moreover, since (7') holds only in steady state, λ is no longer constant through time, but evolves along a transitional path. Furthermore, since debt costs depend upon the stock of debt, the dichotomy of the dynamics characteristic of the simple stationary Ramsey model breaks down. With an upward sloping curve of debt, the dynamics of capital, *K*, its shadow value, *q*, the marginal utility of wealth, λ, and debt, *B*, all become interdependent in a fourth order dynamic system. An example of this has been analyzed by Bhandari, Haque, and Turnovsky (1990), who show that under weak conditions the dynamics are a saddlepath with, *K* and *B* being the two gradually evolving state variables.

Endogenizing debt costs has parallel effects in the other models. Thus, in the basic AK model with fixed employment discussed in Section 4.1, the upward sloping supply curve of debt leads to transitional dynamics. Since in equilibrium the interest rate must be constant, this requires that in the long run B/K must remain constant and this in turn implies that in balanced growth equilibrium foreign assets and domestic capital must grow at the same rate. Since consumption depends upon foreign assets this in turn forces equilibrium consumption to grow at the same rate as domestic capital (and output). The differential growth rates, which we noted could occur with a perfect world capital market are not sustainable indefinitely, since otherwise the country will become an infinite debtor or creditor, continually driving up or down the international borrowing cost. Analytically, we require

$$\psi \equiv \frac{r(B/K)(1-\tau_b)-\rho}{1-\gamma} = \frac{q_1-1}{h} \equiv \varphi \tag{43}$$

where q_1 is determined by (21). These two growth rates, which can diverge indefinitely for a perfect world capital market are now forced into eventual equality through the adjustment in the indebtedness; see Turnovsky (1997b).

The same applies to the non-scale model discussed in Section 5. For precisely the same reason as in the stationary Ramsey model, the transitional dynamics are now augmented. Second, the equilibrium growth rates of consumption, (33'), and output (31), which previously, were allowed to diverge indefinitely in the case of inelastically supplied labor, are now forced into equality. Equation (38), which was introduced as a knife-edge condition in the case of elastic labor supply, now is an equilibrium condition which determines the level of indebtedness; see Eicher and Turnovsky (1999b).

7. Final Comments

In studying the macrodynamics of small open economies particular attention has been paid to ensure that the equilibria are derived as the outcome of a consistently specified intertemporal decision process. In doing this, restrictions must be imposed to ensure that the equilibrium is sustainable. A variety of models have evolved over the years. In this paper we have presented a generic model in sufficient generality to enable us to derive the specific models as special cases.

All macrodynamic models of small open economies impose some such condition. In some cases the conditions imposed are clearly apparent, while other times they are more subtle. But in all cases they affect the dynamic equilibrium and through it the impact of policy. The nature of the knife-edge assumptions typically involves a tradeoff between restrictions on preferences and technology.

Models such as the stationary Ramsey model, discussed in Section 3, or the non-scale growth model with endogenous labor supply, discussed in Section 6.2, which permit the most general technologies, impose the strongest restrictions on preferences. Basically they require that preferences be consistent with the returns on investment, thereby reconciling consumption possibilities with production possibilities.

On the other hand, the standard AK model, by assuming both fixed labor supply and a linear technology in capital, imposes the most stringent technological requirements, but imposes no further restrictions on preferences. The restrictions are dependent on the assumptions about labor supply. In the case that they are elastic, while restrictions remain unrestricted, an appropriate initial exchange of assets is required to ensure that the balanced growth path is consistent with the nation's intertemporal budget constraint.

Finally, we should note some further knife-edge conditions that our analysis has not taken into account. First, the utility function has been assumed to be of the constant utility form, since this generates constant consumption-wealth ratios that are consistent with attaining a steady growth rate. Second, we have assumed that capital accumulation involves adjustment costs. In the absence of such adjustment costs, transitional dynamics in the stationary Ramsey model ceases and the country attain steady state instantaneously. In the case of the basic AK model with fixed labor, to be viable the exogenous fixed return on capital would need to equal the fixed foreign interest rate, leading to a further knife-edge condition.

As briefly discussed in Section 6, the introduction of a limited access to world financial markets, in the form of an upward sloping supply curve of debt breaks the knife-edge condition. There are two ways that this can be viewed. One is that the constraint on borrowing means that the country is no longer small; the very source of the knife-edge we have been emphasizing is thus removed. Another, is that the relationship proxies the cost due to risk. As is well known, the introduction of risk loosens the restrictions imposed by arbitrage conditions, as agents trade off risk and return. One simple way of doing this is through the Blanchard (1985) overlapping generations models. From this standpoint, one may argue it is the assumed absence of risk that is the most stringent knife-edge condition of all.

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