# Demography and Growth: A Unified Treatment of Overlapping Generations

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## Abstract

We construct a unified overlapping-generations (OLG) framework of equilibrium growth, which includes the Blanchard 'perpetual youth" and the Samuelson models as special, polar cases. We derive expressions determining the equilibrium growth rate in labor productivity using a Romer-style production sector for a model with quite general assumptions about mortality, retirement, and other demographic conditions. We numerically determine the equilibrium growth rates for the Blanchard and Samuelson models, and for OLG models intermediate between them, using comparable assumptions to examine how alternative forms of the OLG model can affect the equilibrium growth and savings rates. Models with "realistic" demographic specifications are found to be better approximated by the Samuelson polar case than the Blanchard polar case. We also compare growth rates in OLG models to those of the standard Romer model, which has no significant demographic structure. We examine how the equilibrium growth rate is affected by changes in demographic factors in the different OLG models, including changes in the population growth rate, the ratio of working years to retirement years over the life cycle, and longevity. Growth rates are lower in economies where households work a larger fraction of their lifetimes and where population growth rates are high. In particular, economies with high population growth, short longevity, and limited retirement opportunities occupy a "demographic trap" of low or negative productivity growth rates. Increases in longevity, ceteris paribus, increase the growth rate up to a point, beyond which further increases cause the growth rate to decline.

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## 1. Introduction

Most economists would agree with the view that the demographic structure of an economy is an important determinant of its growth potential and performance. This is manifested in the concern with "ageing economies" and the financing of the impending retirement of the "baby boom" generation. Yet despite the acknowledged importance of these issues, contemporary economic growth theory has not addressed them in any concerted way. The standard benchmark growth models remain based on infinitely-lived representative agent models, which lack the structure necessary to address these intergenerational issues.[1]

Two workhorse models that incorporate demographic features (overlapping generations) have dominated the economics profession: the Samuelson (1958) and the Blanchard (1985) models, both of which appropriately have had profound impacts and provided deep insights.[2] But both are extremely stylized, which limits their applicability to incorporate demographic factors in a comprehensive way. The basic Samuelson model usually assumes a two-period framework – period one for working and period two for retirement – although extensions to an initial third period, for education, also exist.[3] While the Samuelson model enables one to discuss inter-generational issues such as social security, the financing of education, and the national debt, its formulation in terms of discrete time makes it inflexible. For example, as typically specified, the Samuelson model (implicitly) assumes that an agent's working period and retirement period are of equal length (one time unit each) and does not allow for varying lengths of the retirement period relative to the working period, which turns out to be an important issue in many contemporary economies. Also, identifying a time unit with a generation renders the model cumbersome for policy analysis and, as a result, Auerbach and Kotlikoff (1987), in their comprehensive study of fiscal policy, introduced 55 periods so that they could accommodate several generations while employing a plausible time unit.

The Blanchard model is much simpler and amenable to policy analysis in a growth

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[1] We have in mind the Ramsey model or some form of the Romer model, depending upon the underlying production structure.

[2] The Samuelson model is sometimes coupled with Diamond (1965), while Blanchard is often linked with Yaari (1965) and Weil (1985). A comprehensive treatment of the Samuelson and Blanchard models and their applications to issues in economic growth is provided de la Croix and Michel (2002).

[3] See e.g. Docquier and Michel (1999).

framework. But this comes at a price, namely that it assumes a constant mortality hazard function so that the probability of death facing an agent is constant and independent of his age (hence the "perpetual youth" designation.) While this is obviously convenient and provides payoffs in terms of analytical tractability, it is not a good approximation to human mortality data which exhibit age-dependent mortality or senescence (see e.g. Heijdra and Romp (2008), who apply such a model to Dutch data.[4])

In this paper, we construct a unified overlapping generations model of economic growth that includes both the Samuelson and Blanchard models as polar extremes, and equally importantly, admits more plausible demographic structures as intermediate cases. To do this we introduce a mortality function due to de Moivre (1725). Although demographers have traditionally employed the Gompertz (1825) survival function which fits human mortality data well, this function is intractable for analytical purposes. The de Moivre function is more tractable, yet fits the main characteristics of modern human mortality reasonably well, except at the old-age tail of its distribution.[5] A further desirable feature is that it includes the Samuelson and Blanchard models as limiting cases, enabling us to nest the two classic OLG models in a unified framework, together with the conventional representative agent model. We regard the ability to nest the two standard OLG models within a unified structure, rather than presenting them (as is typically done) as alternative approaches, to be significant, in that viewing them in a more general context enhances our understanding of both.[6] In fact, embedding the Samuelson model in this framework increases both its generality and its tractability. It enables us to employ a continuous-time formulation and, more importantly, to analyze working and retirement periods that vary in length, thereby dividing the agent's life into two phases, rather than imposing an inflexible two period time frame.[7]

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[4] Because of its tractability there is a substantial literature introducing the Blanchard mortality structure to growth models; see e.g. Saint Paul (1992), and most recently, Tamai (2009). Bommier and Lee (2003) derive a number of propositions for overlapping generations models with "realistic demography" in exchange economies and in economies without technical progress.

[5] Neither the Gompertz nor the de Moivre survival functions exhibit high infant mortality rates (called the "bath-tub curve" by demographers). However, this phenomenon has largely been eliminated in advanced economies.

[6] For example, in motivating the perpetual youth model as an alternative approach, Blanchard and Fischer (1989, p.115) argue that "overlapping generations models with more than two generations are analytically intractable". .More recently, in introducing the OLG model, Acemoglu (2009, p.327) characterizes the Blanchard perpetual youth model as "a tractable alternative to the basic OLG model."

[7] We should note that there is a literature analyzing continuous-time overlapping generations economies, with finite horizons, that originated with Cass and Yaari (1967). This literature tends to focus on issues related to existence of

In order to focus on the generality of the mortality and demographic factors, we simplify the production side of the economy, and assume that output is produced by a Romer-type production function, whereby the return to capital remains constant and the equilibrium of the economy is always on its balanced growth path. We begin by providing a general characterization of the equilibrium for a general survival function. Using the de Moivre survival function, we parameterize the model so that we can solve explicitly for the equilibrium growth rate in polar cases of the Samuelson and Blanchard specifications and can compare them to the standard Romer case. For intermediate cases, analytical solutions are not possible but we can solve numerically for the equilibrium growth rate for parameter values that approximate reality.

We determine numerically the equilibrium growth rates in the Blanchard, Samuelson, and intermediate models under comparable calibrations and examine how alternative forms of the OLG model affect the magnitude of the equilibrium growth rate, and the response to different demographic conditions. We also compare the growth rates of OLG models to the standard Romer growth model. Demographic factors may increase or decrease the growth rate relative to the standard model. Equilibria of models with "realistic" assumptions regarding mortality are found to be closer to the Samuelson polar case than to the Blanchard polar case. We examine how the equilibrium growth rate is affected by changes in demographic factors in the different OLG models, including changes in the population growth rate, the ratio of working years to retirement years over the life cycle, and longevity. Growth rates are lower in economies where life expectancy is short, households work a large fraction of their lifetimes, and where population growth rates are high. This suggests that such economies may occupy a "demographic trap" of low and negative growth rates. Increases in longevity increase the growth rate up to a point, but further increases can decrease the growth rate. These demographic characteristics raise important policy questions. However, since the objective of the present paper is to introduce a more general demographic structure we do not address them, leaving that for future work.

The remainder of the paper is structured as follows. Section 2 lays out the basic analytical

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equilibrium and its characterization in a more abstract context than we have in mind here; see e.g. Burke (1996), and Edmond (2008).

framework for households, while Section 3 describes the demographic structure, derives descriptive expressions for the aggregate economy, and explains solution methods. Section 4 parameterizes the demographic functions using the de Moivre functional form, and recapitulates the equations of the model. Section 5 performs numerical simulations. These involve deriving the equilibrium growth rate and savings rate for various cases of the de Moivre function, including the polar Samuelson and polar Blanchard as boundary cases. We also conduct some sensitivity analysis in these two cases. Section 6 concludes, while the Appendix explicitly solves the model for the Samuelson and Blanchard cases and shows that both converge to the representative agent model as lives (and working lives) become arbitrarily long.

## 2. The Analytical Framework

In an overlapping generations framework, it is important to distinguish clearly between household age and calendar time.[8] To avoid potential confusion between these two time concepts, we adopt a particular notational convention. Specifically, household variables are indexed in parentheses by age (indexes may be *x, y,* or *z*). Where a variable depends on calendar time, the variable is indexed by the means of subscripts. Thus, for example, $v_t(x)$ denotes the value of variable $v$ at time $t$ for a household of age $x$. When household indexes are absent, the time subscript denotes an economy-wide value of the variable at the subscripted time. The current time is denoted $t$, so, for example, $w_t$ denotes the value of the variable $w$ prevailing in the economy at the current time. The absence of an age index always indicates that the variable is independent of age. Aggregate variables, by summing over cohorts, depend on calendar time only.

### 2.1 Households

We let $S(z)$ denote the probability at birth of surviving to age $z$ and denote the maximum attainable age as $\omega$. Because the probability of survival declines with age, $S'(z) < 0$, $0 < z < \omega$, with $S(z) = 0$, for $z \geq \omega$. With this notation, $S(z)/S(x)$ is the probability of surviving to age $z$ conditional on surviving to age $x$, while $-S'(z)/S(z)$ is the mortality hazard rate at age $z$.

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[8] In the standard infinitely-lived representative agent framework these concepts coincide, so that the distinction is irrelevant.

We shall focus initially on a household of age *x*, at time *t*, and assume that this household maximizes its expected utility over the remainder of its life, that is:

$$U_t(x) = \int_{z=x}^{z=\omega} \frac{S(z)}{S(x)} \cdot e^{-\rho\cdot(z-x)} \cdot u(c_t(z)) \cdot dz \tag{1a}$$

where $\rho$ is the pure time discount rate, and $c_t(z)$ is its planned consumption at age *z*. The household's flow budget constraint at age *z* is

$$\frac{df_t(z)}{dz} \equiv f_t'(z) = i(z)\cdot f_t(z) + w_t(z)\cdot L(z) - c_t(z) \tag{1b}$$

where $w_t(z)$ is the market wage facing the household when it is age *z*, $L(z) \le 1$ is the fraction of the household's unit time endowment supplied as labor at age *z*, $f_t(z)$ is the household's financial wealth, and $i(z)$ is the interest rate on financial wealth facing the household when it is age *z*.[9] The budget constraint (1b) can be expressed equivalently

$$\frac{d}{dz}\left(R(z,x)\cdot f_t(z)\right) = R(z,x)\cdot\left[w_t(z)\cdot L(z) - c_t(z)\right] \tag{1b'}$$

where $R(z,x) \equiv e^{-\int_{y=x}^{y=z} i(y)\cdot dy}$ is the discount factor for a flow at age *z* to a household at age *x*.

Defining the present value Hamiltonian

$$H_t(z) \equiv e^{-\rho\cdot(z-x)}\frac{S(z)}{S(x)}\left\{u(c_t(z)) + \phi_t(z)\cdot\left[w_t(z)\cdot L(z) - c_t(z) - i(z)\cdot f_t(z)\right]\right\} \tag{2}$$

and optimizing with respect to $c_t(z)$ and $f_t(z)$, we obtain the first order conditions

$$u'(c_t(z)) = \phi_t(z) \tag{3a}$$

$$\rho - \frac{\phi_t'(z)}{\phi_t(z)} - \frac{S'(z)}{S(z)} = i(z)\,. \tag{3b}$$

Equation (3a) equates the marginal utility of consumption to the shadow value of financial wealth,

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[9] We assume that $L(x)$ is specified exogenously. While the function can be quite general, for simplicity we assume that it does not vary with (calendar) time.

while (3b) equates the rate of return on consumption, modified by the mortality hazard rate, to the rate of return on financial assets. In addition, the agent must satisfy the transversality condition, which for the agent having a maximum lifespan of $\omega$ is[10]

$$R(\omega, x).f_t(\omega) = 0 \tag{3c}$$

Henceforth, we assume an iso-elastic utility function of the form $u\left(c_t\left(z\right)\right) = \frac{1}{\varepsilon}c_t\left(z\right)^{\varepsilon}$ ($\varepsilon \leq 1$) where $\frac{1}{1-\varepsilon}$ is the intertemporal elasticity of substitution. This enables us to rewrite (3b) as

$$\frac{c_t'\left(z\right)}{c_t\left(z\right)} = \frac{1}{\left(1-\varepsilon\right)} \cdot \left(i\left(z\right) - \rho + \frac{S'\left(z\right)}{S\left(z\right)}\right). \tag{4}$$

We follow Blanchard (1985) and Yaari (1965) in assuming that when mortality hazard is present, households invest wholly in actuarially fair life annuities, so that

$$-\frac{\partial R\left(z,x\right)/\partial z}{R\left(z,x\right)} = i\left(z\right) = r - \frac{S'\left(z\right)}{S\left(z\right)} \tag{5}$$

where *r* is the risk-free rate of return on capital, and $-S'\left(z\right)/S\left(z\right)$ is the mortality hazard premium for a household at age *z*. We assume that *r* is constant, an assumption that is duly validated under the assumption of a Romer (1986) endogenous growth technology. Combining (4) and (5) yields

$$\frac{c_t'\left(z\right)}{c_t\left(z\right)} = \frac{1}{\left(1-\varepsilon\right)} \cdot \left(r - \rho\right) \tag{6}$$

Equation (6) expresses how household consumption changes with its age. This equation will be recognized as the fundamental growth equation in the standard endogenous growth model, based on the infinitely-lived representative agent model, where it is identified with the equilibrium consumption growth rate over time.[11] As our analysis will emphasize below, this does not coincide with the aggregate equilibrium growth rate in an economy with heterogeneous cohorts. Integrating (6), we can express the agent's consumption level at age *z* (relative to that at age *x*) in the form

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[10] As $\omega \to \infty$, the transversality condition converges to the conventional expression $\lim_{\omega \to \infty} R(\omega, x).f_t(\omega) = 0$.

[11] See e.g. Romer (1986). But it also describes the equilibrium consumption growth rate in the two-sector Lucas (1988) model, as developed by Bond, Wang, and Yip (1996).

$$c_t(z) = c_{t-x}(x) \cdot e^{\frac{r-\rho}{1-\varepsilon}\cdot(z-x)}. \tag{7}$$

To express this in terms of the agent's financial resources, we proceed as follows. First, integrating (5) yields

$$R(z,x) = e^{-r(z-x)} \cdot \frac{S(z)}{S(x)}. \tag{5'}$$

Second, integrating (1b') forward at age *z* and using the transversality condition, (3c), we obtain the agent's intertemporal budget constraint applicable from age *x*

$$\int_{z=x}^{z=\omega} R(z,x) \cdot c_t(z)\,dz = f_t(x) + \int_{z=x}^{z=\omega} R(z,x) \cdot w_t(z) \cdot L(z) \cdot dz \tag{8}$$

Finally, substituting for (7) and (5') into (8) and evaluating, we may express the agent's consumption at age *x* in the form

$$c_t(x) = m(x) \cdot v_t(x) \tag{9a}$$

where $v_t(x)$ denotes the "all-inclusive" wealth of a household of age *x* at time *t*, defined as

$$v_t(x) \equiv f_t(x) + h_t(x) = f_t(x) + \int_{z=x}^{z=\omega} e^{-r\cdot(z-x)} \cdot \left( \frac{S(z)}{S(x)} \right) \cdot w_t(z) \cdot L(z) \cdot dz \tag{9b}$$

and $m(x)$ denotes the household marginal propensity to consume out of current all-inclusive wealth, defined by

$$m(x) = \left[ \int_{z=x}^{z=\omega} e^{\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\cdot(z-x)} \cdot \left( \frac{S(z)}{S(x)} \right) \cdot dz \right]^{-1}. \tag{9c}$$

That is, at age *x* a household spends a fraction $m(x)$ of its all-inclusive wealth, $v_t(x)$, consisting of its financial assets, $f_t(x)$, plus its human wealth, $h_t(x)$, where human wealth is equal to the present value of the household's expected future labor income. Note that both human wealth, $h_t(x)$, and the marginal propensity to spend out of wealth reflect the agent's expected mortality. A uniform increase in expected longevity, as represented by an increase in $(S(z)/S(x))$, leads to a ceteris paribus increase in human wealth but a lower propensity to consume. Henceforth, the term "wealth" will refer to all-inclusive wealth with $f_t(\cdot)$ distinguished as financial wealth where it is needed for clarification.

The productivity of labor is assumed to increase over time at a constant rate $g$ (to be determined in equilibrium). Thus the current market wage at time $t$, $w_t = w_{t-x} \cdot e^{g \cdot x}$ where $w_{t-x}$ is the wage rate prevailing in the economy at the time the household is born. This market wage is economy-wide and common to all agents, irrespective of their birth date. Substituting this into $h(x)$, defined in (9b), the human wealth of a household of age $x$ at time $t$ can be expressed as

$$h_t(x) = w_t \cdot p(x) \tag{10a}$$

$$p(x) = \int_{z=x}^{z=\omega} e^{-(r-g)\cdot(z-x)} \cdot \left( \frac{S(z)}{S(x)} \right) \cdot L(z) \cdot dz\ . \tag{10b}$$

Written in this way we see that the human wealth of an agent equals the current wage rate scaled by the present value factor $p(x)$ (which is independent of $t$) , which reflects the discounted future labor supply, adjusted for the rate of productivity growth and the agent's probability of survival.

From (10a) and (10b), the human wealth of an $x$ year old agent at birth is

$$h_{t-x}(0) = w_{t-x} \cdot p(0)\,,$$

where

$$p(0) = \int_{z=0}^{z=\omega} e^{-(r-g)\cdot z} \cdot S(z) \cdot L(z) \cdot dz\ . \tag{10b'}$$

If we assume, further, that every household begins with no financial wealth (no inheritance), the overall wealth at birth of a household currently aged $x$ consists entirely of its initial human wealth, namely

$$v_{t-x}(0) = h_{t-x}(0) = w_{t-x} \cdot p(0)\,.$$

Combining (9a) and (7) enables us to write

$$v_t(x) = c_t(x) \cdot m(x)^{-1} = c_{t-x}(0) \cdot e^{\left(\frac{r-\rho}{1-\varepsilon}\right)\cdot x} \cdot m(x)^{-1}\,. \tag{11a}$$

With no initial financial wealth, (9a) further implies that for this cohort

$$c_{t-x}\left(0\right)=m\left(0\right)\cdot h_{t-x}\left(0\right)=m\left(0\right)\cdot p\left(0\right)\cdot w_{t-x}$$

which when combined with the previous equation yields

$$v_t\left(x\right)=\frac{m\left(0\right)}{m\left(x\right)}\cdot p\left(0\right)\cdot w_{t-x}\cdot e^{\left(\frac{r-\rho}{1-\varepsilon}\right)\cdot x}=\frac{m\left(0\right)}{m\left(x\right)}\cdot p\left(0\right)\cdot w_t\cdot e^{\left(\frac{r-\rho}{1-\varepsilon}-g\right)\cdot x} \tag{11b}$$

where setting $x=0$ in (9c) we have

$$m\left(0\right)=\left[\int_{z=0}^{z=\omega} e^{\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\cdot z}\cdot S\left(z\right)\cdot dz\right]^{-1}. \tag{11c}$$

## 3. The Aggregate Economy

To derive the aggregate economy we need to describe its demographic structure. Let $B_{t-x}$ denote the size of the cohort born at time *t-x*. Given the mortality function, $S(x)$, the current size of that cohort at time *t*, (now of age *x*), $N_t(x)$, is $N_t\left(x\right)=B_{t-x}\cdot S\left(x\right)$. If we assume that birth cohorts grow at rate $n$ over time, then $B_t=B_{t-x}\cdot e^{n\cdot x}$ and we may express $N_t(x)$ in terms of the size of the current birth cohort:

$$N_t\left(x\right)=B_t\cdot e^{-n\cdot x}\cdot S\left(x\right). \tag{12}$$

Aggregating over all cohorts, the total population size at time *t* is[12]

$$N_t=\int_{x=0}^{x=\omega} N_t\left(x\right)\cdot dx=B_t\cdot\Sigma^N,\quad \text{where } \Sigma^N\equiv\int_{x=0}^{x=\omega} e^{-n\cdot x}\cdot S\left(x\right)\cdot dx. \tag{12'}$$

Given the time-invariance of the mortality function, (12’) implies that total population grows at the same rate, *n*, as does the birth cohort.

The current number of deaths of persons of age *x* is $D_t\left(x\right)=-B_t\cdot e^{-n\cdot x}\cdot S'\left(x\right)$ so the total

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[12] Whenever an aggregate variable is obtained by summing over households of all ages in the population, the resulting age and time-independent coefficient will be denoted by $\Sigma$, superscripted by the corresponding variable.

number of current deaths equals $D_t = B_t \cdot \Sigma^D$ where $\Sigma^D \equiv -\int_{x=0}^{x=\omega} e^{-n\cdot x} \cdot S'(x) \cdot dx$. This can be integrated by parts to obtain $D_t = B_t - n \cdot N_t$. Rearranging, population growth $n \cdot N_t$ equals births $B_t$ less deaths $D_t$, with the latter also growing over time at rate *n*.

## 3.1 The Aggregate Household Sector

We now use the demographic structure to obtain the aggregate variables, namely total labor supply, labor income, wealth components, and consumption.

(i) **Total labor supply**, $L_t$, at time *t* is obtained by aggregating the labor supply across all cohorts. That is,

$$L_t = \int_{x=0}^{x=\omega} N_t(x) L(x) dx = B_t \int_{x=0}^{x=\omega} e^{-n\cdot x} \cdot S(x) L(x) dx = B_t \cdot \Sigma^L \tag{13a}$$

$$\Sigma^L \equiv \int_{x=0}^{x=\omega} e^{-n\cdot x} \cdot S(x) \cdot L(x) \cdot dx. \tag{13b}$$

$\Sigma^L$ is the ratio of the labor supply to the size of the birth cohort time *t*.

(ii) **Aggregate labor income** earned at the current time, obtained by aggregating over all cohorts (and taking account of their respective survival rates), is given by

$$Y_t^L = \int_{x=0}^{x=\omega} N_t(x) \cdot L(x) \cdot w(x) \cdot dx.$$

Since all agents are paid the same prevailing wage at time *t*, $w(x) = w_t$, we can write

$$Y_t^L = w_t \cdot B_t \cdot \Sigma^L = w_t \cdot L_t \tag{13c}$$

From (13c) we see that $Y_t^L$ grows at rate $n + g$, the sum of the growth rates of the labor supply $L_t$ and labor productivity, as reflected in the growth rate of the wage rate.

(iii) Using equation (10a), (12), and (13c), we can express **aggregate human wealth**, $H_t$, as

$$H_t \equiv \int_{z=0}^{z=\omega} N_t\left(x\right)\cdot h_t\left(x\right)\cdot dx = w_t B_t \int_{z=0}^{z=\omega} e^{-n\cdot x}\cdot S\left(x\right)\cdot p(x)\cdot dx = Y_t^L \cdot \frac{\Sigma^H}{\Sigma^L} \tag{14a}$$

$$\Sigma^H \equiv \int_{x=0}^{x=\omega} e^{-n\cdot x}\cdot S\left(x\right)\cdot p\left(x\right)\cdot dx\,. \tag{14b}$$

(iv) Combining (11b) and (12), we can express aggregate **all-inclusive wealth**, $V_t$, in the form

$$V_t \equiv \int_{x=0}^{x=\omega} N_t\left(x\right)\cdot v_t\left(x\right)\cdot dx = B_t\cdot w_t\cdot p\left(0\right)\cdot m\left(0\right)\cdot \int_{x=0}^{x=\omega} e^{\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\cdot x}\cdot S\left(x\right)\cdot m\left(x\right)^{-1}\cdot dx\,.$$

which recalling the definition of $m(x)$ can be expressed as

$$V_t = Y_t^L \cdot \frac{p\left(0\right)\cdot m\left(0\right)\cdot \Sigma^V}{\Sigma^L} \tag{15a}$$

$$\Sigma^V \equiv \int_{x=0}^{x=\omega} e^{\left(r-g-n\right)\cdot x}\cdot \int_{z=x}^{z=\omega} e^{\left(\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\right)\cdot z}\cdot S\left(z\right)\cdot dz\cdot dx\,. \tag{15b}$$

Since $V_t = F_t + H_t$, where **aggregate financial wealth** $F_t \equiv \int_{x=0}^{x=\omega} N_t\left(x\right)\cdot f\left(x\right)\cdot dx$, we can write

$$F_t = Y_t^L \cdot \frac{p\left(0\right)\cdot m\left(0\right)\cdot \Sigma^V - \Sigma^H}{\Sigma^L}\,. \tag{16}$$

We assume the economy is one in which financial wealth is positive, so that $p\left(0\right)\cdot m\left(0\right)\cdot \Sigma^V > \Sigma^H$.[13]

(v) **Aggregate consumption** is $C_t = \int_{x=0}^{x=\omega} N_t\left(x\right)\cdot c_t\left(x\right)\cdot dx$. Substituting equations (11a), (11b), and (12), we obtain

$$C_t = Y_t^L \cdot \frac{p\left(0\right)\cdot m\left(0\right)\cdot \Sigma^C}{\Sigma^L} \tag{17a}$$

$$\Sigma^C = \int_{x=0}^{x=\omega} e^{\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\cdot x} S\left(x\right)\cdot dx\,. \tag{17b}$$

Combining (17a) with (15a), we see that $C_t = \frac{\Sigma^C}{\Sigma^V}\cdot V_t$, so that $\frac{\Sigma^C}{\Sigma^V}$ measures the aggregate marginal

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[13] Technically, individual households could hold negative annuity wealth, which would be equivalent to borrowing and buying life-insurance to retire the debt in case of death. We assume that aggregate household annuity wealth is positive.

propensity to consume wealth.

(vi) Finally, we consider the **aggregate accumulation of financial wealth**. We specify this by the quantity

$$\frac{dF_t}{dt} \equiv \int_{x=0}^{x=\omega} N_t(x) \cdot \frac{df_t(x)}{dx} \cdot dx - T_t, \tag{18}$$

which is equal to the increase in financial wealth of households of all ages minus $T_t$, where $T_t$ is the financial wealth transferred from the households who die to the living households (via the actually neutral insurance scheme). Substituting equation (2), we find

$$\frac{dF_t}{dt} = Y_t^F + Y_t^L - C_t$$

where $Y_t^F \equiv \int_{x=0}^{x=\omega} N_t(x) \cdot i(x) \cdot f_t(x) \cdot dx$ denotes total financial income. Using equation (5), we can express $Y_t^F = r \cdot F_t + T_t$ where $T_t = -\int_{x=0}^{x=\omega} N_t(x) \cdot \frac{S'(x)}{S(x)} \cdot f_t(x) \cdot dx = \int_{x=0}^{x=\omega} D_t(x) \cdot f_t(x) \cdot dx$ is the transfer component, $D_t(x)$ is the number of households of age $x$ who die at the current time, and $D_t(x) \cdot f(x)$ is the wealth reclaimed by the annuities company which is used to pay the mortality risk premium to the surviving households. Thus we can now write

$$\frac{dF_t}{dt} = T_t + r \cdot F_t + Y_t^L - C_t - T_t = r \cdot K_t + Y_t^L - C_t. \tag{19}$$

The quantities $\left[ m(0), p(0), \Sigma^C, \Sigma^L, \Sigma^H, \Sigma^V \right]$ which appear in equations (11) through (17) depend upon (i) the constant parameters, $\varepsilon, \rho,$ and $n$, the functional forms of $S(x)$ and the $L(x)$, and the assumed constancy of the growth rate $g$ and the rate of return on capital $r$. As a consequence the aggregate economic variables $\left(V_t, C_t, F_t, H_t, Y_t^L\right)$ grow at the sum of the growth rates of the labor supply and labor productivity, or $n + g$.

## 3.2 The Aggregate Production Sector

In deriving the behavior of the household, the level of the wage rate, the rate of return on capital, and the growth rate of labor productivity have all been assumed to be exogenously given constants, and the equations obtain for any values of $w_t, r,$ and $g$. To complete the model and determine the equilibrium, these need to be derived.

The nature of these equilibrium values will depend upon the underlying production technology. In the case of the Romer production function, with its implied constant productivity of capital (AK) technology, $r$ and g will indeed be constant, consistent with the assumption we have been imposing.[14]

Specifically, we assume that there are $L_t$ identical firms, and each hires one unit of labor and $k_t$ units of capital. Each firm produces output, denoted by $q_t$, in accordance with the Romer-type (1986) Cobb-Douglas production function[15]

$$q_t = A \cdot k_t^{a} \cdot \left( \frac{K_t}{L_t} \right)^{1-a} - \delta \cdot k_t$$

net of capital replacement which depreciates at rate $\delta$, where $A$ is the total factor productivity term, $K_t$ denotes the aggregate capital stock, and $K_t/L_t$ is the economy-wide capital-labor ratio. This last term provides the production externality that ensures that the equilibrium productivity of capital remains constant, thereby enabling the economy to sustain a constant equilibrium growth rate.[16]

Assuming each firm is small enough to ignore its own impact on the economy-wide values of $K_t$ and $L_t$, and since firms are identical, so that in equilibrium $k_t = K_t/L_t$, the equilibrium rate of return on capital given by its marginal product is

$$r = \alpha \cdot A - \delta \tag{20a}$$

which is constant over time. Further, aggregate output, $Q_t$, has the "AK" technology, $Q_t = L_t \cdot q_t = \left(A - \delta\right) \cdot K_t$, while the equilibrium wage rate is given by

$$w_t = q_t - \left(\alpha \cdot A - \delta\right) \cdot k_t = \left(1 - \alpha\right) \cdot A \cdot \frac{K_t}{L_t}. \tag{20b}$$

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[14] If the underlying production function is neoclassical, these quantities would be time varying and this would need to be taken into account by the household sector in its decision-making process.

[15] While Romer (1986) specified the production function as Cobb-Douglas, the crucial properties we obtain apply to any linearly homogeneous production function of the form $f(k_t, K_t/L_t)$.

[16] Turnovsky (2009) reviews and summarizes alternative production externality models used in growth theory.

Since financial capital is just the claim on the physical capital in the economy, $F_t \equiv K_t$. Substituting (20a) and (20b) into (19) this equation reduces to the aggregate goods market clearing condition

$$\frac{dK_t}{dt} = (A-\delta)K_t - C_t \tag{21}$$

Since $K_t \equiv F_t$ grows at rate $g+n$ while $L_t$ grows at the population growth rate *n*, the wage rate, $w_t$, grows at the constant rate *g* over time, thereby validating the assumption we have imposed.

In summary, the aggregate economy is always on its balanced growth path. This is not surprising given (i) that the underlying technology is of the AK form with only one capital stock, (ii) the constant elasticity utility function, and (iii) the time invariance with respect to the mortality hazard and labor supply functions. Changes in any of these aspects will generate transitional dynamics.[17]

### 3.3 Closing the System

The final step to characterize the equilibrium is to determine the equilibrium growth rate *g*. In the conventional Romer model, this is obtained directly from the goods market equilibrium condition

$$\frac{\dot{K}_t}{K_t} = (A-\delta) - \frac{C_t}{K_t} \tag{21'}$$

and substituting for $C_t/K_t$. The same procedure can be performed here, although the equation is highly nonlinear in the growth rate, raising the issue of multiple solutions. Substituting for $\dot{K}_t/K_t$ and the expressions for the aggregate economy summarized in Sections 3.1-3.3, we can write this as

$$g + n = r + A(1-\alpha)\left[1 - m(0)p(0)\frac{\Sigma^C}{\Sigma^L}\right] \tag{22}$$

which in turn can be expressed in terms of the underlying demographic characteristics in the form

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[17] For example, Tamai (2009) introduces public and private capital, which introduces transitional dynamics.

$$g+n=r+A(1-\alpha)\left[1-\left(\frac{\int_{x=0}^{x=\omega} e^{-(r-g)\cdot x}\cdot S(x)\cdot L(x)\cdot dx}{\int_{x=0}^{x=\omega} e^{-n\cdot x}\cdot S(x)\cdot L(x)\cdot dx}\right)\left(\frac{\int_{x=0}^{x=\omega} e^{\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\cdot x} S(x)\cdot dx}{\int_{x=0}^{x=\omega} e^{\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\cdot x}\cdot S(x)\cdot dx}\right)\right] \quad (22')$$

It is readily apparent from (22’) that $g=r-n$ is a solution for any arbitrary mortality hazard function, $S(x)$, and labor supply function, $L(x)$. Rewriting (21’) in the form

$$g+n=r+\frac{1}{K_t}\left[w_t L_t - C_t\right] \quad (21'')$$

we see that this occurs if and only if $C_t = w_t L_t$. In the Appendix we show that this “solution” is incompatible with the individual’s intertertemporal budget constraint. In effect, it violates the agents’ transversality conditions and therefore can be ruled out as nonviable. In Section 4 below we shall plot (22’) and solve it numerically for various forms of mortality functions.

Further insight is obtained by viewing the equilibrium from the standpoint of the equilibrium factor shares. This approach requires the equilibrium growth rate to equate the relative factor share of capital to labor income as determined by aggregate household accumulation to the factor income share as determined in production. While this does not provide any additional information in a representative agent economy, it may do so in the OLG framework.

Focusing on household accumulation, we find that the implied shares of capital income to labor income is

$$\left(\frac{r\cdot K_t}{w_t\cdot L_t}\right)^H = \frac{r\cdot F_t}{Y_t^L} = \frac{r\cdot\left[p(0)\cdot m(0)\cdot\Sigma^V - \Sigma^H\right]}{\Sigma^L}, \quad (23a)$$

whereas the share of capital to labor income determined by production is

$$\left(\frac{r\cdot K_t}{w_t\cdot L_t}\right)^P = \frac{\alpha\cdot A-\delta}{(1-\alpha)\cdot A}. \quad (23b)$$

Equating (23a) and (23b)we find that the equilibrium growth rate, *g*, must satisfy the condition

$$\frac{\left[p(0)\cdot m(0)\cdot\Sigma^V - \Sigma^H\right]}{\Sigma^L} = \frac{1}{(1-\alpha)\cdot A} \quad (23c)$$

where the numerator of the left hand side depends on the growth rate *g*. Thus, (23c) implicitly solves for the equilibrium growth rate as a function of the parameters of the model.

Equations (22’) and (23c) must yield the same viable equilibrium growth rate. While (22’) is simpler to solve, it yields a “solution” $g = r - n$, which as we noted we rule out as nonviable. On the other hand, (23c) involves the expression $\Sigma^V$, defined in (15b) as a double integral, which is correspondingly more difficult to evaluate. Moreover, the fact that $g = r - n$ does not solve (23c), is further evidence that this “solution” can be dismissed as being extraneous. We use both methods in order to simplify the solution algorithm and to rule out solutions that imply counterfactual equilibrium conditions, such as negative financial wealth.

## 4. Parameterization of Survival and Labor Supply Functions

In order to evaluate the quantities $\left[m(0), P(0), \Sigma^C, \Sigma^H, \Sigma^V, \Sigma^L\right]$ appearing in the expressions characterizing the macroeconomic equilibrium (and particularly the growth rate) we need to parameterize the functional forms $S(x)$ and the $L(x)$ which determine how survival and labor supply vary with age. In the Blanchard model, households possess “perpetual youth”, so the survival function takes the single parameter exponential form $S(z) = e^{-\theta \cdot x}$ where $-S'(z)/S(z) = \theta$ is the mortality hazard rate and $1/\theta$ is the life-expectancy of a household of any age. In reality, of course, households possess senescence or aging. It is well known that variants of the famous two parameter Gompertz (1825) survival function fits human mortality data quite well, but the function is intractable for modeling purposes.[18]

One hundred years before Gompertz, de Moivre (1725) observed that “…the number of lives existing at any age is proportional to the number of years intercepted between the age given and the extremity of old age”. This suggests a simple two parameter survival function which can be written in the form

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[18] The Gompertz log linear mortality hazard function implies a two parameter survival function of the Gumbel extreme value form. Specifically, $S(x) = e^{\frac{a}{b}\cdot\left(1-e^{b\cdot x}\right)}$ where $a, b > 0$ are parameters. The exponential survival function is a limiting case of the Gompertz survival function when $b \to 0$.

$$S\left(x\right)=\left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1} \tag{24a}$$

where $\omega$ is the maximum possible age to which a household can possibly live and $1\leq\omega\cdot\theta$ (that is, $1/\theta\leq\omega$).[19]

The de Moivre function is reasonably tractable yet fits the main features of modern human mortality passably well. Most importantly from our standpoint this function allows the Blanchard (1985) "perpetual youth" model and the Samuelson (1958) "one-hoss shay" model (certain survival to a maximum age as in the famous poem by Oliver Wendell Holmes Sr.[20]) as limiting, polar cases. In particular, the limit of expression (24a) as $\omega\rightarrow\infty$ is Blanchard's exponential survival function $S\left(z\right)=e^{-\theta\cdot x}$.[21] The Samuelson model is obtained by setting $\theta\cdot\omega=1$. The household survives with certainty until age $\omega$, at which time it dies with certainty.[22] If $\theta\cdot\omega>\left(<\right)2$, the survival function is strictly concave (convex) as shown in Figure 1.

The Figure shows the two polar cases, Samuelson and Blanchard, and two intermediate cases. In intermediate case 1, it is assumed that life expectancy at birth is equal to half the longest possible life span while in intermediate case 2 it is assumed that life expectancy at birth is equal to 2/3 of the longest possible life span.[23] The survival function for intermediate case 2 is the closest approximation to an actual survival function in a developed economy among those in the de Moivre class.[24]

Analogously we assume that individual labor supply follows a similar process

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[19] Originally, de Moivre proposed a linear function, where $\theta\omega=2$, which is one of the special cases we shall compute numerically in Section 5. The de Moivre and the Gompertz survival functions are nested in the function $S\left(x\right)=\exp\left\{\left(\frac{\theta\omega-1}{1-\beta\omega}\right)\cdot\left[\left(1-\frac{x}{\omega}\right)^{1-\beta\omega}-1\right]\right\}$ which converges to the de Moivre as $\beta\omega\rightarrow1$ and the Gompertz as $\omega\rightarrow\infty$ (see Kohler and Kohler (2000).)

[20] "Have you heard of the wonderful one-hoss shay, That was built in such a logical way, It ran a hundred years to a day…"

[21] The limiting cases as $\omega\rightarrow\infty$ can be evaluated using $\lim\limits_{\omega\rightarrow\infty}\left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega}=e^{-\theta\cdot x}$

[22] The Samuelson model is typically presented in discrete time rather than the continuous time of this paper. As will be seen, the continuous time version is more tractable.

[23] Life expectancy at birth is about 80 years in developed economies, and the documented longest life is 122 years, so we treat this as the most realistic case.

[24] Actual survival functions are convex except at extreme old age.

$$L(x)=\left(1-\frac{x}{\ell}\right)^{\lambda\cdot\ell-1} \tag{24b}$$

with limiting cases: i) $\lim_{\ell\to\infty} L(x)=e^{-\lambda\cdot x}$ and ii) the limit as $\lambda\cdot\ell\to 1$ which implies that $L(x)=1$ for $0\le x<\frac{1}{\ell}$ and $L(x)=0$ otherwise.[25] The first limiting case assumes that the fraction of a household's time devoted to labor declines exponentially with age as in the generalization of the Blanchard model developed by Blanchard and Fischer (1989).[26] In other words, the household retires "gradually", reducing its labor supply at rate $\lambda\cdot e^{-\lambda\cdot x}$ at age *x*. The second case assumes that households supply one unit of labor until a given retirement age $\ell\le\omega$, and zero units thereafter. The second case matches a Samuelson model where households live and work for an interval $(0,\ell)$, retire with certainty at age $\ell$, and supply no labor over the remaining interval $(\ell,\omega)$.

### 4.2 Summarized General and Parameterized Models

We summarize the aggregate economy with both its general and parameterized forms in equation (25). The equilibrium growth rate can be found using the goods market clearing condition by substituting (25b)-(25f) into (25a) and solving for *g*. Alternatively, the equilibrium growth rate can be found using the factor shares method by substituting (25b)-(25e) and (23g)-(25h) into (25a') and solving for *g*.

$$g+n=\frac{S}{K}=r+A\cdot(1-\alpha)\left[1-\cdot\frac{p(0)\cdot m(0)\cdot\Sigma^{C}}{\Sigma^{L}}\right] \tag{25a}$$

$$\frac{\left[p(0)\cdot m(0)\cdot\Sigma^{V}-\Sigma^{H}\right]}{\Sigma^{L}}=\frac{1}{(1-\alpha)\cdot A} \tag{25a'}$$

$$r=\alpha\cdot A-\delta \tag{25b}$$

$$p(0)=\int_{x=0}^{x=\omega} e^{(g-r)\cdot x}\cdot S(x)\cdot L(x)\cdot dx=\int_{x=0}^{x=\omega} e^{(g-r)\cdot x}\cdot\left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1}\cdot\left(1-\frac{x}{\ell}\right)^{\lambda\cdot\ell-1}\cdot dx \tag{25c}$$

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[25] A useful extension would be to derive the labor supply function from underlying behavioral considerations.

[26] In his original paper, Blanchard assumed that labor earnings are constant over time, while Blanchard and Fischer (1989) assumed that they declined with age. Since labor productivity is grows at rate *g* with time, labor earnings change with age at rate *g*-*λ*.

$$m\left(0\right)^{-1}=\int_{x=0}^{x=\omega}e^{\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\cdot x}S\left(x\right)\cdot dx=\int_{x=0}^{x=\omega}e^{\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\cdot x}\cdot\left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1}\cdot dx \tag{25d}$$

$$\Sigma^{L}=\int_{x=0}^{x=\omega}e^{-n\cdot x}\cdot S\left(x\right)\cdot L\left(x\right)\cdot dx=\int_{x=0}^{x=\omega}e^{-n\cdot x}\cdot\left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1}\cdot\left(1-\frac{x}{\ell}\right)^{\lambda\cdot\ell-1}\cdot dx \tag{25e}$$

$$\Sigma^{C}=\int_{x=0}^{x=\omega}e^{\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\cdot x}S\left(x\right)\cdot dx=\int_{x=0}^{x=\omega}e^{\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)\cdot x}\cdot\left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1}\cdot dx \tag{25f}$$

$$\begin{aligned}\Sigma^{V}&=\int_{x=0}^{x=\omega}e^{\left(r-g-n\right)\cdot x}\cdot\int_{z=x}^{z=\omega}e^{\left(\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\right)\cdot z}\cdot S\left(z\right)\cdot dz\cdot dx\\&=\int_{x=0}^{x=\omega}e^{\left(r-g-n\right)\cdot x}\cdot\int_{z=x}^{z=\omega}e^{\left(\frac{\varepsilon\cdot r-\rho}{1-\varepsilon}\right)\cdot z}\cdot\left(1-\frac{z}{\omega}\right)^{\theta\cdot\omega-1}\cdot dz\cdot dx\end{aligned} \tag{25g}$$

$$\begin{aligned}\Sigma^{H}&=\int_{x=0}^{x=\omega}e^{-\left(g+n-r\right)\cdot x}\cdot\int_{z=x}^{z=\omega}e^{\left(g-r\right)\cdot z}\cdot S\left(z\right)\cdot L\left(z\right)\cdot dz\cdot dx\\&=\int_{x=0}^{x=\omega}e^{-\left(g+n-r\right)\cdot x}\cdot\int_{z=x}^{z=\omega}e^{\left(g-r\right)\cdot z}\left(1-\frac{z}{\omega}\right)^{\theta\cdot\omega-1}\cdot\left(1-\frac{z}{\ell}\right)^{\lambda\cdot l-1}\cdot dz\cdot dx\end{aligned}. \tag{25h}$$

The above equations determine the growth rate. Using the equilibrium $g$, the state of the aggregate economy is summarized by the following equations:

$$\frac{Y_{t}^{L}}{B_{t}}=w_{t}\cdot\Sigma^{L} \tag{26a}$$

$$\frac{V_{t}}{Y_{t}^{L}}=\frac{p\left(0\right)\cdot m\left(0\right)\cdot\Sigma^{V}}{\Sigma^{L}} \tag{26b}$$

$$\frac{C_{t}}{V_{t}}=\frac{\Sigma^{C}}{\Sigma^{V}} \tag{26c}$$

$$\frac{H_{t}}{Y_{t}^{L}}=\frac{\Sigma^{H}}{\Sigma^{L}} \tag{26d}$$

$$\frac{F_{t}}{Y_{t}^{L}}=\frac{K_{t}}{Y_{t}^{L}}=\frac{1}{\left(1-\alpha\right)\cdot A} \tag{26e}$$

$$\text{savings rate }\equiv\frac{Sav_{t}}{Q_{t}}=1-\left(\frac{(1-\alpha)A}{A-\delta}\right)\frac{\Sigma^{C}}{\Sigma^{L}}p(0)m(0) \tag{26f}$$

$$w_{t}=\left(1-\alpha\right)\cdot A\cdot\frac{K_{t}}{L_{t}} \tag{26g}$$

In general, we are unable to obtain closed form solutions for the macroeconomic equilibrium described by (25) and (26). The two exceptions are for the two polar cases of the Blanchard “perpetual youth” model and the Samuelson “one-hoss shay” model, the solutions for which are set out in Appendix A.2. There we also show that assuming a stationary population ($n = 0$) and letting $(\lambda, \theta) \to 0$ in the Blanchard model or letting $(l, \omega) \to \infty$ in the Samuelson model, the equilibrium growth rate reduces to the standard Romer result, $g = (r - \rho)/(1 - \varepsilon)$.

## 5. Numerical Comparisons of Blanchard, Samuelson and Intermediate Case

In general, to proceed further we must employ numerical simulations. To do this, we begin by establishing a benchmark specification that can be used to compared the outcomes in the Blanchard, Intermediate, and Samuelson cases of the general OLG model, and compare these OLG outcomes to the outcomes in the standard representative agent growth model. Since the production and preference characteristics of the economy are standard and well documented in the literature, we maintain the following values throughout our analysis. Specifically, on the production side we set $A = 0.35, \alpha = 1/3, \delta = 0.05$, which implies a return to capital $r = 0.067$. Regarding preferences, we set $\rho = 0.03$, $\varepsilon = -1.5$ (i.e. intertemporal elasticity of substitution 0.4).[27]

In contrast, we vary extensively the demographic characteristics of the model. While we take the population growth rate of $n = 0.015$ as a benchmark, we allow it to vary between 0 (stationary population) and 0.03, which we characterize as a high population growth rate. We assume that life expectancy when the household enters the economy (equal to $\int_{x=0}^{x=\omega} S(x) \cdot dx = \int_{x=0}^{x=\omega} \left(1 - \frac{x}{\omega}\right)^{\theta \cdot \omega - 1} \cdot dx$) is 60 years and that the household expects to work 2/3 of this expected life span and expects to be retired for the final 1/3.[28] This requires us to vary the parameters of the household’s age-dependent labor supply function $L(x) = \left(1 - \frac{x}{\ell}\right)^{\lambda \cdot \ell - 1}$. In the benchmark specification, we adjust the values of

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[27] The only point to note is that $\rho$, being a pure rate of time preference, is somewhat smaller than the conventional value for the representative agent model ($\rho = 0.04$). This is because the latter implicitly discount for mortality factors, which we are explicitly incorporating in our analysis.

[28] We assume the household enters the economy as an adult of 20 years so that life expectancy at birth is approximately 80 years. Since childhood is ignored in this model, we refer to the time the household enters the economy and the time of its birth interchangeably.

$l, \omega, \lambda,$ and $\theta$ (depending on the model) to ensure that this "working time ratio" holds. In the Samuelson model, this simply requires that we set $l = 40, \omega = 60$. In the Blanchard model, it requires that we set $\theta = 1/60$ and $\lambda = \cdot\theta/2 = 1/120$. More generally, we set

$$\int_{x=0}^{x=\ell} \left(1 - \frac{x}{\ell}\right)^{\lambda\cdot\ell-1} \cdot \left(1 - \frac{x}{\omega}\right)^{\theta\cdot\omega-1} = wtr \cdot \int_{x=0}^{x=\omega} \left(1 - \frac{x}{\omega}\right)^{\theta\cdot\omega-1} \quad . \tag{27}$$

where "wtr" denotes the working time ratio.[29]

### 5.1 The Effect of Model Structure

Table 1A shows the growth and saving rates in four versions of the general OLG model, including the Blanchard and Samuelson polar cases for economies with population growth rates of 1.5%. In all cases, households expect to be retired for the latter third of their lives.[30] Except in the Samuelson case, households may retire "gradually", withdrawing their labor in accordance with the parameterized labor supply function. The observed differences in growth and savings rates reflect only the differences in the mortality and labor supply functions. In the Blanchard exponential case, the economy-wide saving rate (as a percentage of net output) is 9.3% and the labor productivity growth rate is 1.29% At the opposite pole, the Samuelson model, the economy-wide saving rate is 11.66% and labor productivity growth rate is 2%. Not surprisingly, the outcomes of the intermediate cases lie in between. For the "linear" intermediate case ($\lambda \cdot \ell = \theta \cdot \omega = 2$, see Figure 1), the economy saves 9.79% and labor productivity grows at 1.44%. In the convex (square-root) case ($\lambda \cdot \ell = \theta \cdot \omega = 3/2$), the economy saves 10% and labor productivity grows at 1.52%.

In the convex case, the upper bound on age $(\omega)$ is 50% greater than life expectancy at birth, which accords with factual mortality in developed economies. Since, however, the square-root assumption is perhaps more applicable to mortality than to retirement behavior, we also show in

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[29] We use the FindRoot utility in *Mathematica* to solve equations (25b)-(25f) and (25a) simultaneously. (The same solution can be found using the factor shares method.) The *Mathematica* code is available from the authors on request. In the Blanchard and Samuelson cases, the program finds the roots of explicit equations shown in Appendix 2. In the intermediate cases, the program finds roots of integral equations, which proves to be quite challenging. Although multiple roots are possible, we can eliminate roots with negative financial wealth. In all cases, only a single root is found in the relevant range.

[30] Except in the Samuelson case, this expectation includes the fact that households might die while working.

Table 1a the results for mixed intermediates cases where we set $\lambda \cdot \ell = 1$ so that people retire at a fixed age as in the Samuelson model. This raises the savings and growth rates in both intermediate models. In the square-root case, the economy saves 11.08% and grows at 1.82%. Thus a realistic (i.e., convex) mortality specification coupled with a fixed retirement age results in outcomes that are close to those observed in the polar Samuelson model.

Figure 3 plots the savings-capital ratio, $Sav/K$ and the growth of capital $\dot{K}/K$ in the case where $\theta \cdot \omega = 3/2,\ \lambda \cdot l = 1$. The savings rate has an inverted-U shape as the productivity growth increases, $g$, increases. For low growth rates an increase in productivity has significant positive effects on the return to capital providing agents an incentive to save and accumulate more capital. However, as the growth rate increases and capital become more abundant, its productivity declines and the incentive to save declines correspondingly.[31] We have also plotted these two relationships for other specifications of the demographic functions. They are similar and are not reported.

In Table 1b, we set the population growth rate equal to zero so that the results for OLG models, where households are mortal and retire later in life, can be compared to the traditional Romer growth model where households are identical, infinitely lived and do not retire. As seen in the table, the outcomes in the Romer model lie outside the OLG set, with lower saving and growth rates. Thus, for the benchmark specification, the introduction of OLG demographic realities, such as finite lives and retirement, increases saving and growth relative to the standard growth model.[32] Not surprisingly, the difference is greatest for the Samuelson model where the saving and growth rates are 74% higher than in the standard growth model.

Increased convexity of the survival function means increased certainty regarding the length of life. The convexity of the survival function has increased significantly for developed economies over the past hundred years (see Figure 2 for the United States). Table 1 demonstrates that this secular trend for greater certainty about length of life increases the growth rate for our benchmark specification.

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[31] The two curves intersect at two points (i) $g = 0.052$, which is non-viable, and (ii) at the true equilibrium $g = 0.0182$.

[32] However, as seen in the next section, demographic considerations can result in lower growth rates, even negative growth rates.

## 5.2 The Effects of Population Growth and Retirement

Table 2 shows the effects of population growth rates and the length of working and retirement periods on growth and saving. To simplify the table we show only the polar Blanchard and Samuelson cases for stable population, medium population growth, and high population growth economies. The static population Romer growth rate is shown for comparison. We consider different working and retirement period length scenarios in terms of the values of $l$ and $\omega$.[33] Moving to the right across row increases longevity whereas moving down in any column increases the fraction of expected life that is worked.

Moving across any row, we see that increasing the longevity, holding constant the number of years worked, uniformly increases the labor productivity growth and savings rates. Moving down any column, holding longevity constant, we observe that increasing the number of working years uniformly decreases labor productivity growth and savings rates. This reflects the fact that life-cycle motives for saving are increased in both cases, and higher saving leads to higher growth rates [34]. By comparing the tables, we see that increasing the population growth rate, ceteris paribus, decreases the labor productivity growth rate uniformly, even as it increases saving rates. Faster growing populations mean higher saving rates because they contain higher proportions of young, saving households. However, higher population growth dilutes the effect of capital accumulation on increasing labor productivity. In all cases, of course, saving and growth rates are higher in the Samuelson model than the Blanchard model.

Highest growth rates are found in economies where populations are stable or grow at moderate rates, and which have developed to the stage where households expect to spend a significant fraction of their lives in retirement, and save accordingly. Low productivity growth rates, even negative rates, are found in economies with high population growth rates, short life spans, and where households work most of their lives. Negative productivity growth rates can occur because negative saving by older, richer, less numerous cohorts outweighs the saving of younger, poorer, more numerous cohorts. This scenario is suggestive of a negative or slow growth demographic trap.

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[33] In the Blanchard case, we set $\theta = 1/\omega$ and $\lambda = 1/\ell$.

[34] In this paper, we have not introduced the institution of Social Security.

Declining standards of living can lead to negative aggregate saving in OLG models, and high population growth rates reduce the effect of capital accumulation on increasing labor productivity over time.

### 5.3 Increases in Longevity

Table 3 shows the effects on growth rates resulting from increases in longevity, holding other demographic factors constant. In particular, we hold the fraction of an expected lifetime that households spend working (retired) constant. Again, we consider only the polar Blanchard and Samuelson cases, and examine stable, medium and high population growth rate economies. In all cases, we find that increases in longevity initially increase the growth and saving rates, but these rates reach a maximum beyond which higher longevity actually decreases the rates. This non-monotonic relationship between longevity and growth rates is particularly pronounced in the Samuelson model.[35]

A plausible reason for this observed non-monotonic relationship is as follows. Increased longevity increases planned retirement consumption levels because households expect and choose rising consumption levels over their life-cycles, necessitating higher saving rates by young households. This increases saving and growth rates with longevity. However, because the working period is longer than the retirement period, as longevity increases further, the longer working period implied by a constant fraction of the expected lifetime spent working, along with growing labor productivity, allows younger households to save for retirement at a lower rate, thereby decreasing aggregate saving and reducing growth.

## 6. Conclusions

We have developed a tractable, yet realistic, model of equilibrium growth in which overlapping generations of households are born, work and save, retire and die. The model contains the two "classic" OLG models developed by Samuelson and Blanchard as limiting polar cases, and converges to the standard representative agent growth model as we let lifetimes and working

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[35] The inverted-U relation ship between growth and life expectancy is consistent with empirical evidence; see An and Jeon (2006). It has also been obtained by Tamai (2009).

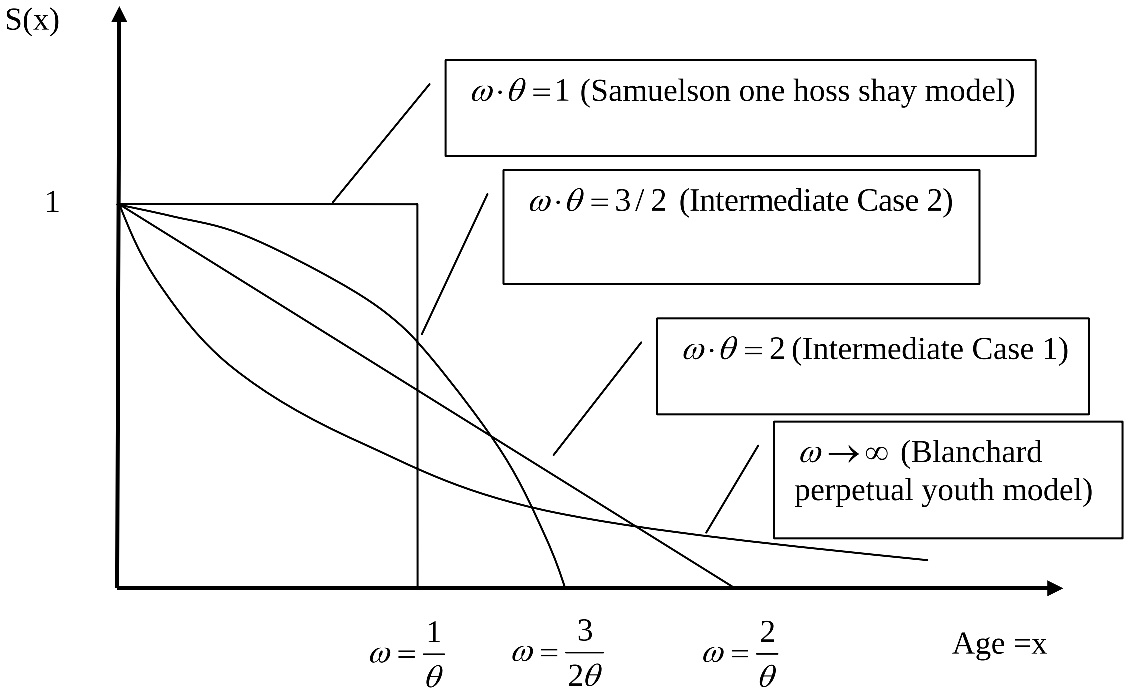
lifetimes increase without bound.

Equilibrium growth rates are calculated for economies that have different demographic conditions in terms of population growth, longevity, and working to retirement periods. In general we find that as the household's horizon is tightened, moving from the standard infinite horizon growth model to the Samuelson model, saving and productivity growth rates are increased. Increasing the fraction of an expected lifetime retired also increases saving and productivity growth rates. Increasing the population growth rate decreases productivity growth rates, even though it increases saving rates. Increasing longevity, holding the fraction of an expected lifetime worked, initially increases productivity growth rates, but increases in longevity beyond a critical level will actually decrease the growth rate. The model suggests a low or negative productivity growth "trap" for economies that have high population growth and rates, and where households have low longevity and work most or all of their expected lifetimes.

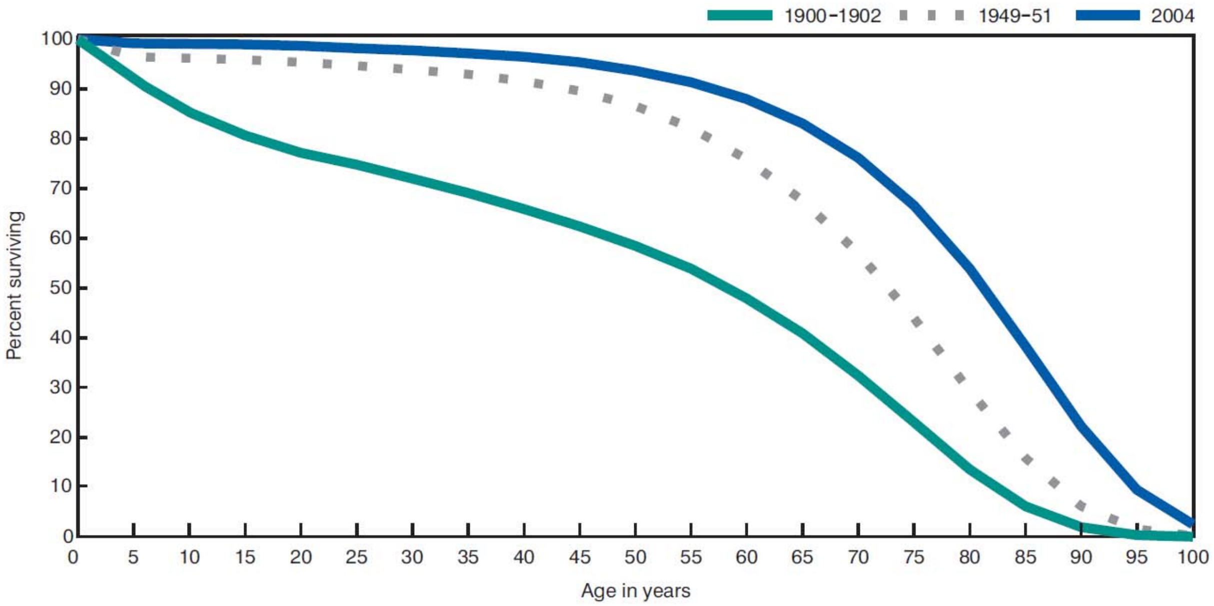
As we stressed at then outset, our concern has been to set out a unified framework for incorporating a plausible demographic structure, within which the Samuelson and Blanchard models can be introduced as limiting cases. The structure we have developed provides a realistic and tractable framework for analyzing all kinds of policy issues pertaining to retirement, social security, and intergenerational transfers. We propose to address some of these in forthcoming work.

Finally, in order to focus on the demographics, we have adopted a simple "AK", production technology, a consequence of which is that the economy is always on its balanced growth path. An important extension of this analysis would be to combine the more general demographic structure we have introduced with a less restrictive neoclassical production function, thereby enabling us to consider the impact of demographic structural changes on the dynamics of the economy as factor returns evolve. We are confident that the approach we have adopted in this paper is an important step toward achieving this more general objective.

## Figure 1: The De Moivre Survival Function

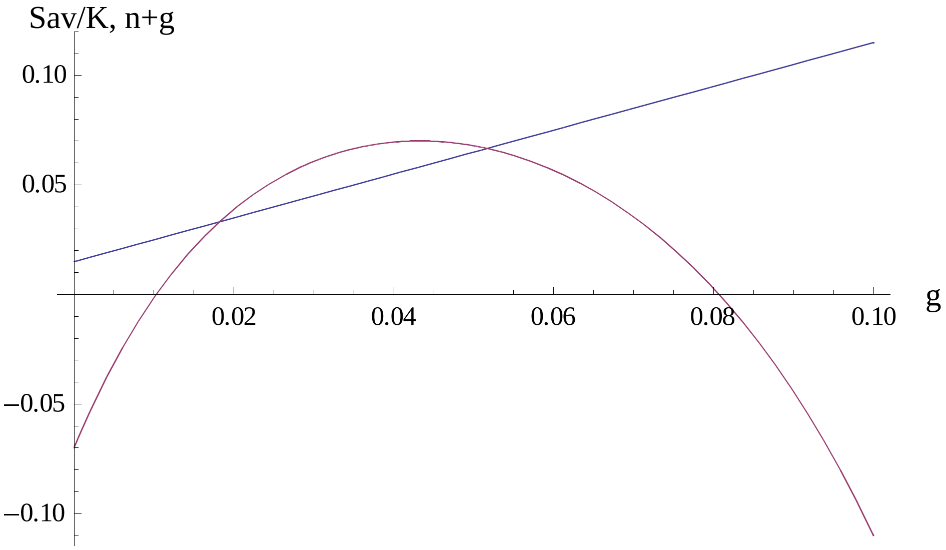


## Figure 2: US Survival Functions, 1900 and 2003



*Source: National Vital Statistics Reports, Vol. 56 #9, December 28 2007.*

## Figure 3: Equilibrium for ω∗θ = 3/2



## Table 1a: Growth-Savings Equilibrium for Basic Parameters (n = 0.015)

| | Blanchard | $\theta\omega=2$ | $\theta\omega=1.5$ | Samuelson |
|---|---|---|---|---|
| | $\lambda=\frac{1}{120},\theta=\frac{1}{60}$ | $\ell=51,\omega=120$ | $\ell=47,\omega=90$ | $\ell=40,\omega=60$ |
| Growth rate | | | | |
| $\lambda\cdot\ell=\theta\cdot\omega$ | 1.29 | 1.44 | 1.52 | 2.00 |
| $\lambda\cdot\ell=1$ | | 1.69 | 1.82 | |
| Savings rate | | | | |
| $\lambda\cdot\ell=\theta\cdot\omega$ | 9.30 | 9.79 | 10.00 | 11.66 |
| $\lambda\cdot\ell=1$ | | 10.65 | 11.08 | |

## Table 1b: Growth-Savings Equilibrium for Stationary Population (n = 0)

| | Romer | Blanchard | $\theta\omega=2$ | $\theta\omega=1.5$ | Samuelson |
|---|---|---|---|---|---|
| | $l\to\infty,\omega\to\infty$ | $\lambda=\frac{1}{120},\theta=\frac{1}{60}$ | $l=51,\omega=120$ | $l=47,\omega=90$ | $l=40,\omega=60$ |
| Growth rate | | | | | |
| $\lambda\cdot\ell=\theta\cdot\omega$ | 1.47 | 1.69 | 1.77 | 1.90 | 2.55 |
| $\lambda\cdot\ell=1$ | | | 2.35 | 2.44 | |
| Savings rate | | | | | |
| $\lambda\cdot\ell=\theta\cdot\omega$ | 4.89 | 5.65 | 5.89 | 6.34 | 8.51 |
| $\lambda\cdot\ell=1$ | | | 7.82 | 8.14 | |

Parameter values: $A=0.35,\alpha=1/3,\rho=0.03,\delta=0.05,\varepsilon=-1.5$
Demographic parameters have been chosen so that in all cases the ratio of agents expected working horizon to lifetime is 2/3.

Table 2

Stable population: $n = 0, A = 0.35$

| | $\omega = 20$ | | | | $\omega = 40$ | | | | $\omega = 60$ | | | | $\omega \to \infty, l \to \infty$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Samuelson | | Blanchard | | Samuelson | | Blanchard | | Samuelson | | Blanchard | | Romer | |
| | growth | saving | growth | saving | growth | saving | growth | saving | growth | saving | growth | saving | growth | saving |
| $l = 20$ | -9.04 | -30.12 | -0.22 | -0.73 | 5.72 | 19.08 | 2.56 | 8.55 | 8.54 | 28.48 | 3.50 | 11.68 | 1.47 | 4.89 |
| $l = 40$ | -- | -- | -- | -- | -1.56 | -5.18 | 0.81 | 2.69 | 2.55 | 8.51 | 1.69 | 5.65 | | |
| $l = 60$ | -- | -- | -- | -- | -- | -- | -- | -- | -0.01 | -0.02 | 1.07 | 3.58 | | |

Medium population growth rate: $n = 0.015, A = 0.35$

| | $\omega = 20$ | | | | $\omega = 40$ | | | | $\omega = 60$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Samuelson | | Blanchard | | Samuelson | | Blanchard | | Samuelson | | Blanchard | |
| | growth | saving | growth | saving | growth | saving | growth | saving | growth | savings | growth | savings |
| $l = 20$ | -9.26 | -25.85 | -0.80 | 2.35 | 4.84 | 21.11 | 2.07 | 11.91 | 7.51 | 30.04 | 3.05 | 15.15 |
| $l = 40$ | -- | -- | -- | -- | -1.69 | -6.31 | 0.37 | 6.24 | 2.00 | 11.66 | 1.29 | 9.30 |
| $l = 60$ | -- | -- | -- | -- | -- | -- | -- | -- | -0.12 | 4.60 | 0.68 | 7.28 |

High population growth rate $n = 0.03, A = 0.35$

| | $\omega = 20$ | | | | $\omega = 40$ | | | | $\omega = 60$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Samuelson | | Blanchard | | Samuelson | | Blanchard | | Samuelson | | Blanchard | |
| | growth | saving | growth | saving | growth | saving | growth | saving | growth | saving | growth | saving |
| $l = 20$ | -9.51 | -21.70 | -1.41 | 5.31 | 3.95 | 23.17 | 1.55 | 15.16 | 6.50 | 31.67 | 2.55 | 18.51 |
| $l = 40$ | -- | -- | -- | -- | -1.87 | 3.78 | -0.10 | 9.68 | 1.45 | 14.83 | 0.86 | 12.85 |
| $l = 60$ | -- | -- | -- | -- | -- | -- | -- | -- | -0.28 | 9.05 | 0.27 | 10.89 |

Table 3: Increase in longevity

$$A = 0.35$$

| | $n = 0$ | | | | $n = 0.015$ | | | | $n = 0.03$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Samuelson | | Blanchard | | Samuelson | | Blanchard | | Samuelson | | Blanchard | |
| | growth | saving | growth | saving | growth | saving | growth | saving | growth | saving | growth | saving |
| $l = 20, \omega = 30$ | 1.60 | 5.32 | 1.63 | 5.43 | 0.91 | 8.04 | 1.11 | 8.69 | 0.22 | 10.73 | 0.55 | 11.85 |
| $l = 30, \omega = 45$ | 2.56 | 8.53 | 1.70 | 5.68 | 1.95 | 11.51 | 1.26 | 9.20 | 1.35 | 14.50 | 0.78 | 12.62 |
| $l = 40, \omega = 60$ | 2.55 | 8.51 | 1.69 | 5.64 | 2.00 | 11.66 | 1.29 | 9.30 | 1.45 | 14.83 | 0.86 | 12.86 |
| $l = 50, \omega = 75$ | 2.40 | 8.00 | 1.67 | 5.58 | 1.88 | 11.26 | 1.29 | 9.31 | 1.37 | 14.55 | 0.88 | 12.94 |

# Appendix

## A.1 Elimination of Solution $g = r - n$

Summing over surviving members of each cohort, aggregate consumption is

$$C_t = \int_{x=0}^{x=\omega} N(x)c(x)dx = \int_{x=0}^{x=\omega} N(x)\left\{ w_t L(x) + i(x) f(x) - \frac{df(x)}{dx} \right\} dx$$

$$= w_t L_t + \int_{x=0}^{x=\omega} N(x)\left\{ i(x) f(x) - \frac{df(x)}{dx} \right\} dx \qquad \text{(A.1)}$$

From (21”), if $g = r - n$, then $C_t = w_t L_t$, so that (A.1) reduces to

$$\int_{x=0}^{x=\omega} N(x)\left\{ i(x) f(x) - \frac{df(x)}{dx} \right\} dx = 0 \qquad \text{(A.2)}$$

Substituting (12) yields

$$\int_{x=0}^{x=\omega} S(x)e^{-nx}\left\{ i(x) f(x) - \frac{df(x)}{dx} \right\} dx = 0 \qquad \text{(A.3)}$$

Integrating an individual agent’s budget constraint (1b) over his lifetime, recognizing that his initial financial wealth is zero, and taking account of the transversality condition (3c), yields the intertemporal constraint (8), which we may rewrite as

$$\int_{x=0}^{x=\omega} S(x)e^{-rx}\left\{ w(x)L(x) - c(x) \right\} dx = 0 \qquad \text{(A.4)}$$

so that the agent’s present value of consumption equals the present value of his labor income. Substituting (1b) into (A.4) implies

$$\int_{x=0}^{x=\omega} S(x)e^{-rx}\left\{ i(x) f(x) - \frac{df(x)}{dx} \right\} dx = 0 \qquad \text{(A.5)}$$

Equations (A.3) and (A.5) can hold simultaneously if and only if $r = n$. But since for the present AK technology these are independently set parameters, there is no reason for this constraint to hold and hence $g = r - n$ is not a viable equilibrium.

## A.2 Special Cases

In this Appendix, we present the equations for the two polar limiting cases of the general model and demonstrate that both models converge to the standard Romer growth model as lifetimes become infinite.

### A.2.1 The Blanchard Perpetual Youth OLG Model

Taking the limit as $\omega, \ell \to \infty$ in equations (25c) to (25h) and carrying out the integration we obtain

$$g+n=\frac{S}{K}=r+A\cdot(1-\alpha)\cdot\left[1-\frac{p(0)\cdot m(0)\cdot\Sigma^{C}}{\Sigma^{L}}\right] \tag{A.6a}$$

$$\frac{p(0)\cdot m(0)\cdot\Sigma^{V}-\Sigma^{H}}{\Sigma^{L}}=\frac{1}{(1-\alpha)\cdot A} \tag{A.6a'}$$

$$r=\alpha\cdot A-\delta \tag{A.6b}$$

$$p(0)=\frac{1}{r+\theta+\lambda-g} \tag{A.6c}$$

$$m(0)=\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}+\theta \tag{A.6d}$$

$$\Sigma^{L}=\frac{1}{n+\theta+\lambda} \tag{A.6e}$$

$$\Sigma^{C}=\frac{1}{g+n-\left(\frac{r-\rho}{1-\varepsilon}\right)} \tag{A.6f)}$$

$$\Sigma^{V}=\frac{1}{\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}+\theta\right)\cdot\left(g+n+\theta-\frac{r-\rho}{1-\varepsilon}\right)} \tag{A.6g}$$

$$\Sigma^{H}=\frac{1}{(\lambda+n+\theta)\cdot(\lambda+\theta+r-g)} \tag{A.6h)}$$

Substituting (A.6b)-(A.6e) and (A.6g)-(A.6h) into (A.6a'), we obtain the rate of growth of labor productivity in the perpetual youth economy.[1] The labor productivity growth can be expressed as the solution to the quadratic form

$$\left(g-\frac{r-\rho}{1-\varepsilon}\right)^2-\left[(1-\alpha)\cdot A+\lambda+\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}-n\right]\cdot\left(g-\frac{r-\rho}{1-\varepsilon}\right)+\lambda\cdot(1-\alpha)\cdot A-(n+\theta)\cdot\left[\lambda+\theta+\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right]=0 \qquad \text{(A.7)}$$

Taking the negative root, the solution is

$$g=\frac{r-\rho}{1-\varepsilon}+\frac{1}{2}\cdot\left\{\left[(1-\alpha)\cdot A+\lambda+\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}-n\right]-\sqrt{\left[(1-\alpha)\cdot A+\lambda+\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}-n\right]^2+4\cdot\nabla}\right\} \qquad \text{(A.8)}$$

where $\nabla=(n+\theta)\cdot\left[\lambda+\theta+\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right]-\lambda\cdot(1-\alpha)\cdot A$. As the economy converges to the case of a constant population of infinitely lived households who each supply a unit of labor, i.e., $(n,\lambda,\theta)\to 0$, then $\nabla\to 0$ and $g$ converges to the standard expression $\frac{r-\rho}{1-\varepsilon}$. More generally, $g\overset{>}{\underset{<}{-}}\frac{r-\rho}{1-\varepsilon}$ and $\nabla\overset{<}{\underset{>}{-}}0$.

### A.2.2 The Samuelson One-Hoss Shay Model

In this case, $\lambda\cdot\ell=1$ so $\left(1-\frac{x}{\ell}\right)^{\lambda\cdot\ell-1}=1$ for $x\in[0,\ell]$ and zero for $x>\ell$. Similarly, $\left(1-\frac{x}{\omega}\right)^{\theta\cdot\omega-1}=1$ for $x\in[0,\omega]$ and zero for $x>\theta$. Integrating we obtain

$$p(0)=\frac{1-e^{-(r-g)\cdot\ell}}{r-g} \qquad \text{A.6c')}$$

$$m(0)=\frac{\rho-\varepsilon\cdot r}{(1-\varepsilon)\cdot\left(1-e^{-\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right)\cdot\omega}\right)} \qquad \text{(A.6d')}$$

$$\Sigma^L=\frac{1-e^{-n\cdot\ell}}{n} \qquad \text{(A.6e'))}$$

$$\Sigma^C=\frac{1-e^{-\left(\frac{r-\rho}{1-\varepsilon}-g-n\right)}}{g+n-\frac{r-\rho}{1-\varepsilon}} \qquad \text{(A.6f')}$$

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[1] We solve using the factor shares method because in this case the savings equal investment condition, (A.6a) yields a cubic equation.

$$\Sigma^V = \frac{1-\varepsilon}{\rho-\varepsilon\cdot r}\cdot\left(\frac{1-e^{-\left[g+n-\frac{r-\rho}{1-\varepsilon}\right]\cdot\omega}}{\left(g+n-\frac{r-\rho}{1-\varepsilon}\right)} - \frac{\left(e^{-\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\cdot\omega} - e^{-\left(g+n+\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}-r\right)\cdot\omega}\right)}{(g+n-r)}\right) \tag{A.6g'}$$

$$\Sigma^H = \frac{1}{r-g}\cdot\left(\frac{1-e^{-n\cdot\ell}}{n} + \frac{\left(e^{-n\cdot\ell}-e^{-(r-g)\cdot\ell}\right)}{(g+n-r)}\right) \tag{A.6h'}$$

Substituting into (A.6a') and rearranging we get

$$\frac{1}{(1-\alpha)\cdot A} = \frac{1-e^{-(r-g)\cdot\ell}}{\left(1-e^{-\left(\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\right)\cdot\omega}\right)}\cdot\left(\frac{1-e^{-\left[g+n-\frac{r-\rho}{1-\varepsilon}\right]\cdot\omega}}{\left(g+n-\frac{r-\rho}{1-\varepsilon}\right)} - \frac{\left(e^{-\frac{\rho-\varepsilon\cdot r}{1-\varepsilon}\cdot\omega} - e^{-\left(g+n-\frac{r-\rho}{1-\varepsilon}\right)\cdot\omega}\right)}{(g+n-r)}\right)\cdot\frac{n}{(r-g)\cdot\left(1-e^{-n\cdot\ell}\right)}$$
$$-\left(\frac{1-e^{-n\cdot\ell}}{n} + \frac{\left(e^{-n\cdot\ell}-e^{-(r-g)\cdot\ell}\right)}{(g+n-r)}\right)\cdot\frac{n}{(r-g)\cdot\left(1-e^{-n\cdot\ell}\right)}$$

This equation implicitly solves for the labor productivity growth rate. In the “long horizon” case, we take the limit as $\ell,\omega\to\infty$ to obtain $\left[g-\frac{r-\rho}{1-\varepsilon}\right]\cdot\left[\frac{1}{(1-\alpha)\cdot A}+\frac{1}{r-g}\right]=\frac{-n}{(1-\alpha)\cdot A}$.

The transversality condition requires $r>g$, so $g\le\frac{r-\rho}{1-\varepsilon}$ as $n\ge 0$. As expected, in the Samuelson model, the labor productivity growth rate converges to the standard result in the constant, infinitely-lived population case. If the population is growing, the labor productivity growth rate is less than $\frac{r-\rho}{1-\varepsilon}$ in the long-horizon case.

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